REPARAMETRIZATIONS IN STRING FIELD THEORY

By

R. RAJU VISWANATHAN

A DISSERTATION PRESENTED
TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1989

To My Parents

ACKNOWLEDGEMENTS

Many people have assisted me in the completion of this work. First and foremost, I am extremely grateful to my thesis advisor Professor Pierre Ramond from whom I have learnt so much in the past few years. His collaboration, encouragement, patience and timely advice have been invaluable. He has always followed my work closely while at the same time affording me a considerable amount of independance, a method of education that I have found especially beneficial.

I wish to thank Professors D. Drake, R. Field, P. Sikivie and C. Thorn for serving on my supervisory committee. I am very grateful to Vincent Rodgers, whose lively discussions and collaboration provided a strong impetus to my research in its initial stages. I wish to thank all the other people from whom I have learnt physics: the faculty members of the physics department, particularly those in the particle theory group; the postdoctoral associates in the particle theory group; and especially, all my fellow graduate students from whom I have benefited through many discussions.

It is a pleasure to thank Tom McCarty and especially Gary Kleppe. The many discussions that I have had with them have helped me significantly and my collaboration with Gary has been most fruitful. Their friendship has provided me with considerable moral support and they have always been around to produce the measure of comic relief necessary to provide a pleasant working environment. I am very grateful to both of them. I am also particularly grateful to Chang Sub Kim, whose friendship and moral support I value greatly.

I am especially indebted to my friend and room-mate S. Pushpavanam, whom I forgot to acknowledge in my undergraduate project report. I shall make amends by thanking him here for his support and friendship during the past several years. Special thanks also go to P.C. Pratap and my cousin Ravi Viswanath for their encouragement and friendship. I am thankful to all my friends in Gainesville for making my stay here a pleasant one.

My gratitude to my mother Kalpagam, my father Ramachandran and my sister Deepa Lakshmi is beyond measure. Their support, care and affection have always been a source of strength for me. So also is my gratitude to my cousin Prema Kumar, with whose family I have spent many a pleasant weekend in Jacksonville.

This research was supported in part by the United States Department of Energy under contract No. FG05-86-ER40272 and by the Institute for Fundamental Theory.

TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

REPARAMETRIZATIONS IN STRING FIELD THEORY

By

R. Raju Viswanathan

May 1989

Chairman: Pierre Ramond
Major Department: Physics

String theory has recently been recognized as a viable model for the unification of the fundamental forces in nature. Of particular importance is the fact that closed strings contain the graviton as part of their spectrum and could therefore provide us with a consistent quantum theory of gravity. String field theory is a natural arena to examine the dynamics of strings. After the formulation of a gauge-covariant free closed string field theory, an algebraic approach to string field theory based on reparametrization invariance is discussed.

The basic formalism of the algebraic approach is that of the Marshall-Ramond formulation of string field theory, where strings are studied as one-dimensional spacelike surfaces evolving in time. The formalism is extended to include the bosonized ghost field, yielding an anomaly-free algebra in the process. The analysis is extended to superstrings and representations of the super-reparametrization algebra are detailed.

Invariant operators are constructed from the coordinates and the ghost fields. It is shown that these operators obey an anomalous algebra. In particular, the BRST operator is recovered as the trace of a symmetric spacetime tensor. Alternative representations of the superconformal ghost algebra are

considered, leading to supersymmetric bosonization formulae. Dynamical invariants besides the BRST operator are shown to exist in the superbosonized theory.

CHAPTER 1
INTRODUCTION

String theory[1,2] is the most recent attempt in physics to unify the fundamental interactions of nature. Unification has long been a central goal of physics. It has been known for the past several decades that the correct description of microscopic phenomena is in terms of quantum physics. A consistent quantum theory of gravity, however, has eluded physicists. Strings appear to offer new hope in this direction.[3,4,5,6,7] A remarkable feature of strings is that they actually predict the dimensionality of spacetime.[8,9,10] Superstring theory predicts a ten-dimensional world, as opposed to the four-dimensional physical universe that we inhabit. If the theory is to be realistic, therefore, it should also predict how and why the extra dimensions only appear at very small length scales. It turns out, however, that one can construct several distinct compactification schemes for the extra dimensions which yield realistic particle spectra at low energies.[11] The low energy predictions depend, among other things, on the topology of the particular compactification scheme that is assumed. Not enough is known about the dynamics of strings to tell us what the preferred compactification of the theory really is.

While a fair amount is known about the perturbative aspects[12,10,13,14] of strings, it appears that the important issues of low energy physics and the geometry of spacetime require a good deal of knowledge about non-perturbative aspects of strings before they can be sucessfully dealt with. To resolve these issues requires a more fundamental understanding of strings as building blocks than we have at present. The dynamics of closed strings should presumably determine the geometry of spacetime. String field theory is one of several

methods of study that have been proposed towards a better understanding of the dynamics of string theory.[15,16,17] A proper formulation of string field theory would shed light on issues like compactification if the field equations could be solved. It is therefore crucial to construct a gauge-covariant closed string field theory. It is also important to uncover as much of the algebraic structure of the theory as possible since this could yield significant clues to the dynamics. In this thesis, we shall construct a free closed string field theory as a first step towards understanding the dynamics of closed strings and then examine string field theory from a purely algebraic standpoint. In the algebraic approach, reparametrization invariance is taken to be the fundamental symmetry of the theory. In the process, we shall unearth new invariants in string field theory, apart from recovering the usual BRST formulation.[17,18,19,20,21]

This dissertation is organized as follows, in essentially chronological order. A formulation of free closed string field theory[22] is presented first. The construction of a free string field theory for closed bosonic strings is detailed using the Banks-Peskin[20] language of string fields as differential forms. It is shown that it is necessary to introduce an auxiliary field even at the free level in order to construct a lagrangian that yields gauge-unfixed equations of motion. The gauge covariant equations of motion can be obtained from a gauge-fixed set of equations by the process of successive gauge-unfixing.

Secondly, the role of the reparametrization algebra as a fundamental symmetry in bosonic string field theory is studied. The Marshall-Ramond formulation of string field theory is described classically[16] and the relevant operators are identified. Representations of the reparametrization algebra are discussed together with their composition rules. The relevant operators for a general representation are given. The bosonized ghost field is introduced as a

connection term in the covariant derivative over the space of one-dimensional reparametrizations. The theory is quantized and the anomaly in the algebra is found. The states in the theory are characterized and a normal-ordered invariant dynamical operator is constructed, the BRST charge.

Thirdly, the theory is extended to include super-reparametrizations. The Marshall-Ramond extension to superstrings is described. The algebra of super-reparametrizations is derived and its linear representations are given. It is shown that the doublet representation is the only linear representation consisting completely of covariant fields. Composition rules for products of doublets are given. The ghost doublets are constructed and their structure determines the anomaly in the algebra, which vanishes in the critical dimension, namely ten. The BRST charge is again constructed as an invariant dynamical operator.

Next, the superconformal ghosts are fermionized and a set of ghost doublets is catalogued. Invariant vectors are constructed from these and from the coordinate doublet. New tensor invariants are constructed for the bosonic string and the supersymmetric string. The algebra of these tensors is constructed for the bosonic case and it is found to be anomalous.[23] The non-anomalous part of the algebra is projected out by means of a set of matrices. These invariants raise the possibility of a larger symmetry in the theory.

In the last chapter, an alternative representation for the superconformal ghosts is constructed with the techniques developed so far. We look for dynamical invariant operators that can be constructed as the integral of heavier components of doublets. A new four-parameter family of solutions is found, and the BRST charge is recovered as a particular combination of the four solutions. The presence of scalar dynamical invariants besides the BRST operator

points to a richer structure underlying the superbosonized theory. The picture-changing operator of superstring field theory is also obtained from a general four-parameter class of weight zero operators which change the picture number. The existence of a family of weight-zero operators raises the possibility that there might exist other BRST-invariant picture-changing operators in the superbosonized theory besides the usual one. Such operators would have an important role to play in the description of superstring interactions. Finally, the results obtained are summarized.

CHAPTER 2
CLOSED STRING FIELD THEORY

In this chapter we shall detail the construction of a gauge-covariant free closed string field theory. It has been known for some time now that the open bosonic string has a kinetic operator which is simply the BRST charge of first quantized string theory. Then the free Lagrangian[19,21,17,24,25] for an open string field Φ takes the form $< \Phi|Q|\Phi >$ where one must define an appropriate inner product. This kinetic operator is a bit unusual in that it carries with it a non-zero quantum number, namely the ghost number. The construction of the BRST charge ensures that it carries a ghost number of one. Since the Lagrangian must not carry any quantum number, the physical string field must have ghost number $-1/2$, which it does.

When one looks at closed strings, the closed string field Ψ has a ghost number of -1 associated with it, since it is this choice that reproduces all of the physical state conditions

$$(L_0 + \bar{L}_0 - 2)\phi = 0 \tag{2.1}$$

$$(L_0 - \bar{L}_0)\phi = 0 \tag{2.2}$$

$$L_n = \bar{L}_n = 0 \tag{2.3}$$

starting from

$$Q|\phi >= 0 \tag{2.4}$$

as an equation of motion, when we look at a state $|\phi >$ that is annihilated by the ghost and antighost annihilation operators. Thus one cannot have

a Lagrangian of the above form unless one puts in an appropriate insertion with the correct ghost number. It is necessary to have a Lagrangian which yields gauge-covariant equations of motion since one needs to know the off-shell content of the theory to describe interactions. Since we already know that the gauge-covariant equations of motion are of the form[19,17]

$$Q|\Psi>=0 \tag{2.5}$$

with the gauge invariance(as a consequence of $Q^2 = 0$)

$$|\Psi>\rightarrow|\Psi>+Q|\Lambda> \tag{2.6}$$

we would like to construct a Lagrangian which yields this unconstrained equation.

With this motivation, we shall begin our construction by first reviewing the formalism for open strings. The ghost zero mode algebra

$$\{c_0, b_0\} = 1 \tag{2.7a}$$

$$c_0^2 = b_0^2 = 0 \tag{2.7b}$$

results in a two-dimensional representation[18] of states $|->$ and $|+>$ which satisfy

$$c_0|->=|+> \tag{2.8a}$$

$$b_0|+>=|-> \tag{2.8b}$$

$$c_0|+>=b_0|->=0 \tag{2.8c}$$

The states $|+>$ and $|->$ have the inner product relations

$$<-|+>=1 \tag{2.9a}$$

$$<-|->=<+|+>=0 \tag{2.9b}$$

and have ghost numbers 1/2 and −1/2 respectively. The BRST operator can be expanded in the form

$$Q = Kc_0 - b_0 \downarrow + d + \partial. \tag{2.10}$$

The convenience of this form for the BRST operator is that the ghost and antighost zero modes c_0 and b_0, respectively, have been separated out. The operators ∂ and d are simply the Banks-Peskin cohomology generators[20] of the Virasoro algebra; they contain, among others, terms trilinear in the ghost modes with the zero modes excluded. The operators appearing in Q then satisfy the algebra[20]

$$[K, \downarrow] = [K, d] = [K, \partial] = 0 \tag{2.11}$$

$$[\downarrow, \partial] = [\downarrow, d] = d^2 = \partial^2 = 0 \tag{2.12}$$

$$\{d, \partial\} = K\downarrow \tag{2.13}$$

String fields are then viewed as differential forms in this language. A general string field Φ_n^m has covariant and contravariant indices which simply indicate the number of ghost and antighost oscillators respectively that are associated with it; then Φ_n^m can be expanded in the form

$$\Phi_n^m = \phi(x) c_{\alpha_1}..c_{\alpha_n} b_{\beta_1}..b_{\beta_m} \tag{2.14}$$

where $\phi(x)$ stands for a local field and its associated set of coordinate creation operators. Then Φ_n^m acting on the vacuum produces the states in the theory. From this definition, it follows that $(n-m)$ is the ghost number of the form Φ_n^m. The operators ∂ and d act on contravariant and covariant indices respectively to produce forms with one less contravariant index and one more covariant index respectively. The operator $\downarrow$ acts to change a contravariant index to a covariant one.

Since the vacuum state representation of the ghost zero mode algebra is two-fold degenerate, a general string state $|A>$ of ghost number $-1/2$ can be expanded in the form

$$|A>= \Phi_n^n|-> + S_n^{n+1}|+> \tag{2.15}$$

where a summation over n is implied. The zero-form Φ_0^0 corresponds to the physical string field that satisfies the gauge-fixed equation of motion

$$(L_0 - 1)\Phi = K\Phi = 0 \tag{2.16}$$

subject to the 'physical gauge' condition

$$L_n\Phi = 0 \quad \text{for } n > 0 \tag{2.17}$$

or

$$d\Phi = 0. \tag{2.18}$$

The Lagrangian for free open string field theory can be written as

$$\mathcal{L} =< A|Q|A > . \tag{2.19}$$

The corresponding equations of motion

$$Q|A >= 0 \tag{2.20}$$

then take the form[24,26]

$$K\Phi_n^n + \partial S_n^{n+1} + dS_{n-1}^n = 0 \tag{2.21a}$$

$$d\Phi_n^n + \partial\Phi_{n+1}^{n+1} + \downarrow S_n^{n+1} = 0 \tag{2.21b}$$

at the nth level. The gauge invariance at this level which arises as a consequence of the nilpotency of Q is that these equations are invariant under

$$\delta\Phi_n^n = d\Lambda_{n-1}^n + \partial\Lambda_n^{n+1} + \downarrow\chi_{n-1}^{n+1} \tag{2.22a}$$

$$\delta S_n^{n+1} = -K\Lambda_n^{n+1} + d\chi_{n-1}^{n+1} + \partial\chi_n^{n+2} \tag{2.22b}$$

The gauge parameters Λ_n^{n+1} and χ_n^{n+2} themselves have a further gauge invariance due to the nilpotency of Q; the process continues indefinitely. It is therefore necessary to take an infinite number of levels into account if one wants to completely unfix the gauge. The above infinite set of equations of motion can be compactly summarized as

$$K\Phi + (\partial + d)S^1 = 0 \tag{2.23}$$

$$(\partial + d) + \downarrow S^1 = 0. \tag{2.24}$$

Taken all together, these equations are then the gauge-covariant equations of motion.

We shall demonstrate how these equations can be obtained from the physical gauge by the process of gauge unfixing.[24,26] Starting from

$$K\Phi = 0 \tag{2.25}$$

and

$$d\Phi = 0, \tag{2.26}$$

one makes the gauge change

$$\delta\Phi = \partial\Lambda^1 \tag{2.27}$$

which yields the equations

$$K\Phi + K\partial\Lambda^1 = 0 \tag{2.28}$$

$$d\Phi + d\partial\Lambda^1 = 0 \tag{2.29}$$

or(using $\{d, \partial\} = K\downarrow$)

$$K\Phi + \partial K\Lambda^1 = 0 \tag{2.30a}$$

$$d\Phi - \partial d\Lambda^1 + K{\downarrow}\Lambda^1 = 0 \tag{2.30b}$$

We introduce the Stuckelberg field[27,24,28] with the variation

$$\delta\Phi^1_1 = d\Lambda^1 \tag{2.31}$$

and the subsidiary field S^1 with the variation

$$\delta S^1 = -K\Lambda^1 \tag{2.32}$$

to write the above equations (2.30) as

$$K\Phi + \partial S^1 = 0 \tag{2.33}$$

$$d\Phi + \partial\Phi^1_1 + {\downarrow}S^1 = 0 \tag{2.34}$$

Using the fact that $[\partial, \downarrow] = d^2 = \partial^2 = 0$, we see that these equations are invariant under

$$\delta\Phi^1_1 = d\Lambda^1 + \partial\Lambda^2_1 - {\downarrow}\chi^2 \tag{2.35}$$

$$\delta S^1 = -K\Lambda^1 + \partial\chi^2 \tag{2.36}$$

The equation of motion of the Stuckelberg field follows from its original variation with just Λ^1:

$$\begin{aligned} K\Phi^1_1 &= Kd\Lambda^1 = dK\Lambda^1 \\ &= -dS^1 \end{aligned} \tag{2.37}$$

This relation is of course not invariant under the gauge changes generated by Λ^2_1 and χ^2. From the original variation of the Stuckelberg field, we see that it is constrained by

$$d\Phi^1_1 = 0 \tag{2.38}$$

We can repeat the process starting from the equations (2.37) and (2.38) to obtain

$$K\Phi^1_1 = -dS^1 - \partial S^2_1 \tag{2.39a}$$

$$d\Phi_1^1 + \partial\Phi_2^2 + \downarrow S_1^2 = 0 \tag{2.39b}$$

where Φ_2^2 is the new Stuckelberg field at this level defined by its variation

$$\delta\Phi_2^2 = d\Lambda_1^2 \tag{2.40}$$

and the field S_1^2 is the corresponding subsidiary field defined by its variation

$$\delta S_1^2 = -K\Lambda_1^2 + d\chi^2. \tag{2.41}$$

As before, the equations (2.39) have additional invariances given by

$$\delta\Phi_2^2 = d\Lambda_1^2 + \partial\Lambda_2^3 - \downarrow\chi_1^3 \tag{2.42}$$

$$\delta S_1^2 = -K\Lambda_1^2 + d\chi^2 + \partial\chi_1^3. \tag{2.43}$$

Now the process stabilizes and at the n th level we get the equations (2.21a) with the gauge invariance (2.21b); we repeat these here for convenience:

$$K\Phi_n^n + \partial S_n^{n+1} + dS_{n-1}^n = 0 \tag{2.44}$$

$$d\Phi_n^n + \partial\Phi_{n+1}^{n+1} + \downarrow S_n^{n+1} = 0 \tag{2.45}$$

We will now see that this process of gauge unfixing will be useful for the closed string, where an auxiliary field appears. We shall use the above process to conclude that this auxiliary field contains no propagating degrees of freedom.

The BRST charge of the closed string separates into independent left and right moving pieces. It can be written in the form

$$Q = Kc_1 + \bar{K}c_2 - \downarrow b_1 - \bar{\downarrow} b_2 + d + \bar{d} + \partial + \bar{\partial} \tag{2.46}$$

The left and right moving operators are barred and unbarred respectively, and the left and right moving ghost zero modes have a corresponding subscript of 1 or 2 respectively. The operators satisfy

$$[K, \downarrow] = [K, d] = [K, \partial] = 0 \tag{2.47}$$

$$[\downarrow, \partial] = [\downarrow, d] = d^2 = \partial^2 = 0 \tag{2.48}$$

$$\{d, \partial\} = K\downarrow \tag{2.49}$$

and similarly for the right moving operators. The left and right moving operators commute or anticommute with one another as they are independent. The ghost zero mode algebra

$$\{c_1, b_1\} = \{c_2, b_2\} = 1 \tag{2.50}$$

$$c_1^2 = c_2^2 = b_1^2 = b_2^2 = 0 \tag{2.51}$$

has a standard representation in terms of direct products of open string vacua for the left and right moving sectors, given by the states $|-- >$, $|-+ >$, $|+- >$ and $|++ >$ in an obvious notation. These states have ghost numbers of -1, 0, 0 and 1 respectively. The non-zero inner products are

$$< -+|+- > = < --|++ > = 1 \tag{2.52}$$

and the action of the zero modes on the vacua is given by

$$\begin{aligned}
c_1|-- > &= |+- > \\
c_2|-- > &= |-+ > \\
c_1|-+ > &= |++ > \\
c_2|+- > &= -|++ > \\
b_1|++ > &= |-+ > \\
b_2|++ > &= -|+- > \\
b_1|+- > &= |-- > \\
b_2|-+ > &= |-- >
\end{aligned} \tag{2.53}$$

Physical states in the theory are ghost number minus one states. A general state $|\eta>$ of ghost number minus one can be expanded in the form

$$|\eta>= \phi|-- > +\mu^1|-+ > +\sigma^1|+- > +\nu^2|++ > \tag{2.54}$$

As in the case of the open string, the zero-form ϕ contains the physical propagating degrees of freedom. As stated earlier, it is not possible to construct a diagonal Lagrangian of the form $<\eta|Q|\eta>$ without making a suitable insertion. It is not easy to find a satisfactory insertion. We shall therefore try to construct a non-diagonal Lagrangian by introducing an additional ghost number zero string field. We define such a ghost number zero string field as

$$|\tau>= \gamma^1|++ > +\lambda|-+ > +\omega|+- > +\pi_1|-- > \tag{2.55}$$

The zero-form field ϕ_0^0 in $|\eta>$ is the physical string field and its equations of motion in the physical gauge are

$$(L_0 + \bar{L}_0 - 2)\phi_0^0 = (K + \bar{K})\phi_0^0 = 0 \tag{2.56}$$

$$(L_0 - \bar{L}_0)\phi_0^0 = (K - \bar{K})\phi_0^0 = 0 \tag{2.57}$$

The physical gauge conditions read

$$L_n\phi_0^0 = d\phi_0^0 = 0 \tag{2.58}$$

$$\bar{L}_n\phi_0^0 = \bar{d}\phi_0^0 = 0. \tag{2.59}$$

The equation (2.57) above is actually just a kinematical constraint equation since it does not contain any time derivatives. The other fields in the expansion of $|\eta>$ arise as a consequence of moving out of this gauge. The gauge covariant equations of motion of $|\eta>$, which can be obtained by gauge unfixing, can also be obtained by simply acting Q on it. The resulting equations are

$$(d + \bar{d} + \partial + \bar{\partial})\sigma^1 + K\phi - \bar{\downarrow}\nu^2 = 0 \tag{2.60a}$$

$$(d+\bar{d}+\partial+\bar{\partial})\mu^1+\bar{K}\phi-\downarrow\nu^2=0 \tag{2.60b}$$

$$(d+\bar{d}+\partial+\bar{\partial})\nu^2-\bar{K}\sigma^1-K\mu^1=0 \tag{2.60c}$$

$$(d+\bar{d}+\partial+\bar{\partial})\phi+\downarrow\sigma^1+\bar{\downarrow}\mu^1=0 \tag{2.60d}$$

These equations can be obtained from the Lagrangian[22,29,30,31]

$$\mathcal{L}=<\tau|Q|\eta> \tag{2.61}$$

which has the correct ghost number of zero. The equation of motion of the field $|\tau>$ $(Q|\tau>=0)$ reads in component form

$$(d+\bar{d}+\partial+\bar{\partial})\omega-K\pi_1+\bar{\downarrow}\gamma^1=0 \tag{2.62a}$$

$$(d+\bar{d}+\partial+\bar{\partial})\lambda-\bar{K}\pi_1+\downarrow\gamma^1=0 \tag{2.62b}$$

$$(d+\bar{d}+\partial+\bar{\partial})\gamma^1-\bar{K}\omega+K\lambda=0 \tag{2.62c}$$

$$(d+\bar{d}+\partial+\bar{\partial})\pi_1-\downarrow\omega-\bar{\downarrow}\lambda=0 \tag{2.62d}$$

These equations are invariant under the gauge transformations

$$\delta|\eta>=Q|\Lambda> \tag{2.63}$$

and

$$\delta|\tau>=Q|\tilde{\Lambda}> \tag{2.64}$$

Since the equations of motion of the zero forms ω_0^0 and λ_0^0 involve kinetic terms, there arises the possibility that $|\tau>$ might be a propagating field. However, we note that the kinetic term of ω only involves $\bar{K}$ and that of λ only involves K. This suggests that these equations and the kinetic terms arise purely as a consequence of moving out of a set of gauge-fixed equations. The removal of the $K-\bar{K}$ constraint on the physical field ϕ must correspond to a similar removal of the same constraint on a field in the ghost number zero sector $|\tau>$

to which the physical field couples.[22] So we shall start from the gauge-fixed equations

$$(K - \bar{K})T = 0 \tag{2.65a}$$

$$dT = \bar{d}T = 0 \tag{2.65b}$$

The zero form T is the analogue of ϕ_0^0 in the dual space $|\tau >$. The gauge variation of T is

$$\delta T = (K + \bar{K})\alpha + \partial\rho^1 + \bar{\partial}\rho^{\bar{1}}. \tag{2.66}$$

This gauge variation results in the gauge transformed equations

$$(K - \bar{K})T + (K - \bar{K})[(K + \bar{K})\alpha + \partial\rho^1 + \bar{\partial}\rho^{\bar{1}}] = 0 \tag{2.67a}$$

$$dT + d(K + \bar{K})\alpha + d\partial\rho^1 + d\bar{\partial}\rho^{\bar{1}} = 0 \tag{2.67b}$$

$$\bar{d}T + \bar{d}(K + \bar{K})\alpha + \bar{d}\partial\rho^1 + \bar{d}\bar{\partial}\rho^{\bar{1}} = 0 \tag{2.67c}$$

It is understood here that the barred operators only act on barred indices and similarly for the unbarred operators. We shall write ρ^1 and $\rho^{\bar{1}}$ as

$$\rho^1 = \frac{1}{2}(\Omega^1 + \Lambda^1) \tag{2.68}$$

and

$$\rho^{\bar{1}} = \frac{1}{2}(\Omega^{\bar{1}} + \Lambda^{\bar{1}}) \tag{2.69}$$

respectively. The role of the fields Ω^1 and Λ^1 will become clear shortly in the equations which follow. The above gauge transformed equations (2.67) can now be written in the form

$$(K - \bar{K})T + (K + \bar{K})\bar{T} + \partial\gamma^1 + \bar{\partial}\gamma^{\bar{1}} = 0 \tag{2.70a}$$

$$dT + \frac{1}{2}(K + \bar{K})\pi_1 + \frac{1}{2}\downarrow\gamma^1 + \partial T_1^1 + \bar{\partial}T_1^{\bar{1}} = 0 \tag{2.70b}$$

$$\bar{d}T + \frac{1}{2}(K+\bar{K})\pi_{\bar{1}} + \frac{1}{2}\bar{\downarrow}\gamma^{\bar{1}} + \partial T^{1}_{\bar{1}} + \bar{\partial} T^{\bar{1}}_{\bar{1}} = 0 \tag{2.70c}$$

where the variations of the various fields are

$$\delta T = -\frac{1}{2}(K+\bar{K})\alpha + \frac{1}{2}\partial(\Lambda^1 + \Omega^1) + \frac{1}{2}\bar{\partial}(\Lambda^{\bar{1}} + \Omega^{\bar{1}}) \tag{2.71a}$$

$$\delta \bar{T} = \frac{1}{2}(K-\bar{K})\alpha + \frac{1}{2}\partial(\Lambda^1 - \Omega^1) + \frac{1}{2}\bar{\partial}(\Lambda^{\bar{1}} - \Omega^{\bar{1}}) \tag{2.71b}$$

$$\delta\gamma^1 = -K\Lambda^1 + \bar{K}\Omega^1 \tag{2.71c}$$

$$\delta\gamma^{\bar{1}} = -K\Lambda^{\bar{1}} + \bar{K}\Omega^{\bar{1}} \tag{2.71d}$$

$$\delta\pi_1 = d\alpha - \downarrow\Omega^1 \tag{2.71e}$$

$$\delta\pi_{\bar{1}} = \bar{d}\alpha - \bar{\downarrow}\Omega^{\bar{1}} \tag{2.71f}$$

$$\delta T^{\bar{1}}_{1} = \frac{1}{2}d\Omega^{\bar{1}} + \frac{1}{2}d\Lambda^{\bar{1}} \tag{2.71g}$$

$$\delta T^{\bar{1}}_{\bar{1}} = \frac{1}{2}\bar{d}\Omega^{\bar{1}} + \frac{1}{2}\bar{d}\Lambda^{\bar{1}} \tag{2.71h}$$

$$\delta T^{1}_{\bar{1}} = \frac{1}{2}\bar{d}(\Omega^1 + \Lambda^1) \tag{2.71i}$$

$$\delta T^{1}_{1} = \frac{1}{2}d(\Omega^1 + \Lambda^1) \tag{2.71j}$$

The zero form $\bar{T}$ arises as a consequence of unfixing the $(K-\bar{K})$ constraint on T. At this stage it is convenient to introduce a change of variables for T and $\bar{T}$. We define

$$T \equiv \frac{1}{2}(\omega + \lambda) \tag{2.72}$$

and

$$\bar{T} \equiv \frac{1}{2}(\lambda - \omega). \tag{2.73}$$

Similar definitions also hold for the forms T^1_1, etc. This definition enables us to make contact with the BRST equations (2.62). Then ω and λ transform as

$$\delta\omega = K\alpha + \partial\Omega^1 + \bar{\partial}\Omega^{\bar{1}} \tag{2.74}$$

$$\delta\lambda = \bar{K}\alpha + \partial\Lambda^1 + \bar{\partial}\Lambda^{\bar{1}} \tag{2.75}$$

and the equations (2.70) become

$$K\lambda - \bar{K}\omega + \partial\gamma^1 + \bar{\partial}\gamma^{\bar{1}} = 0 \tag{2.76a}$$

$$\begin{aligned} &d\omega + d\lambda + (K + \bar{K})\pi_1 + \downarrow\gamma^1 \\ &\quad + \partial\omega^1_1 + \partial\lambda^1_1 + \bar{\partial}\omega^{\bar{1}}_1 + \bar{\partial}\lambda^{\bar{1}}_1 = 0 \end{aligned} \tag{2.76b}$$

$$\begin{aligned} &\bar{d}\omega + \bar{d}\lambda + (K + \bar{K})\pi_{\bar{1}} + \bar{\downarrow}\gamma^{\bar{1}} \\ &\quad + \partial\omega^1_{\bar{1}} + \partial\lambda^1_{\bar{1}} + \bar{\partial}\omega^{\bar{1}}_{\bar{1}} + \bar{\partial}\lambda^{\bar{1}}_{\bar{1}} = 0 \end{aligned} \tag{2.76c}$$

From the variations defining the fields we get the consistency conditions

$$d\omega - d\lambda + (K - \bar{K})\pi_1 + \partial\omega^1_1 - \partial\lambda^1_1 + \bar{\partial}\omega^{\bar{1}}_1 - \bar{\partial}\lambda^{\bar{1}}_1 - \downarrow\gamma^1 = 0 \tag{2.77a}$$

and

$$\bar{d}\omega - \bar{d}\lambda + (K - \bar{K})\pi_{\bar{1}} + \partial\omega^1_{\bar{1}} - \partial\lambda^1_{\bar{1}} + \bar{\partial}\omega^{\bar{1}}_{\bar{1}} - \bar{\partial}\lambda^{\bar{1}}_{\bar{1}} - \bar{\downarrow}\gamma^{\bar{1}} = 0 \tag{2.77b}$$

The new forms at this level obey the constraint equations

$$d\omega^1_1 = d\omega^{\bar{1}}_1 = \bar{d}\omega^1_{\bar{1}} = \bar{d}\omega^{\bar{1}}_{\bar{1}} = 0 \tag{2.78a}$$

and similarly for λ, as well as

$$\bar{d}\omega^1_1 = -d\omega^1_{\bar{1}} \tag{2.78b}$$

$$\bar{d}\lambda^1_1 = -d\lambda^1_{\bar{1}} \tag{2.78c}$$

$$d\pi_1 - \downarrow\omega^1_1 = 0 \tag{2.78d}$$

$$\bar{d}\pi_{\bar{1}} - \bar{\downarrow}\omega^1_{\bar{1}} = 0 \tag{2.78e}$$

$$\bar{d}\pi_1 - \downarrow\omega^1_{\bar{1}} = 0 \tag{2.78f}$$

$$d\pi_{\bar{1}} - \bar{\downarrow}\omega^{\bar{1}}_1 = 0 \tag{2.78g}$$

The equations (2.78) have the further invariances

$$\delta\hat{\omega}_1^1 = \hat{\partial}\hat{\Omega}_1^2 + K\hat{\alpha}_1^1 - \bar{\downarrow}\mu^2 \tag{2.79}$$

$$\delta\hat{\lambda}_1^1 = \hat{\partial}\hat{\Lambda}_1^2 + \bar{K}\hat{\alpha}_1^1 - \downarrow\hat{\Gamma}^2 \tag{2.80}$$

$$\delta\hat{\pi}_1 = \hat{\partial}\hat{\alpha}_1^1 \tag{2.81}$$

$$\delta\hat{\gamma}^1 = \hat{\partial}\hat{\Gamma}^2 \tag{2.82}$$

Here the hats over the fields stand for all possible combinations(barred and unbarred) of the covariant and contravariant indices. For instance, $\hat{\Gamma}^2$ stands for Γ^{11}, $\Gamma^{\bar{1}1}$, $\Gamma^{1\bar{1}}$ and $\Gamma^{\bar{1}\bar{1}}$. The operators with hats over them stand for barred and unbarred operators which act wherever possible. The process stabilizes and we obtain the n th level equations[22]

$$-K\hat{\pi}_n^{n-1} + \bar{\downarrow}\hat{\gamma}_n^{n+1} + \hat{\partial}\hat{\omega}_n^n + \hat{d}\hat{\omega}_{n-1}^{n-1} = 0 \tag{2.83a}$$

$$-\bar{K}\hat{\pi}_n^{n-1} + \downarrow\hat{\gamma}_n^{n+1} + \hat{\partial}\hat{\lambda}_n^n + \hat{d}\hat{\lambda}_{n-1}^{n-1} = 0 \tag{2.83b}$$

$$-\bar{K}\hat{\omega}_n^n + K\hat{\lambda}_n^n + \bar{\partial}\hat{\gamma}_n^{n+1} + \hat{d}\hat{\gamma}_{n-1}^n = 0 \tag{2.83c}$$

$$\hat{d}\hat{\pi}_n^{n-1} + \hat{\partial}\hat{\pi}_{n+1}^n - \downarrow\hat{\omega}_n^n - \bar{\downarrow}\hat{\lambda}_n^n = 0 \tag{2.83d}$$

These equations have a gauge invariance under the gauge variations

$$\delta\hat{\omega}_n^n = \hat{d}\hat{\Omega}_{n-1}^n + \hat{\partial}\hat{\Omega}_n^{n+1} - \bar{\downarrow}\hat{\Gamma}_{n-1}^{n+1} + K\hat{\alpha}_n^n \tag{2.84a}$$

$$\delta\hat{\lambda}_n^n = \hat{d}\hat{\Lambda}_{n-1}^n + \hat{\partial}\hat{\Lambda}_n^{n+1} - \downarrow\hat{\Gamma}_{n-1}^{n+1} + \bar{K}\hat{\alpha}_n^n \tag{2.84b}$$

$$\delta\hat{\gamma}_n^{n+1} = \hat{d}\hat{\Gamma}_{n-1}^{n+1} + \hat{\partial}\hat{\Gamma}_n^{n+2} - K\hat{\Lambda}_n^{n+1} + \bar{K}\hat{\Omega}_n^{n+1} \tag{2.84c}$$

$$\delta\hat{\pi}_{n+1}^n = \hat{d}\hat{\alpha}_n^n + \hat{\partial}\hat{\alpha}_{n+1}^{n+1} + \bar{\downarrow}\hat{\Lambda}_n^{n+1} + \downarrow\hat{\Omega}_n^{n+1} \tag{2.84d}$$

We have added and subtracted equations (2.77a) and (2.77b) with equations (2.76b) and (2.76c) to obtain equations (2.83a) and (2.83b). The equations

(2.83) of course are just the equations of motion (2.62) of the dual field $|\tau>$. If one can reverse the above process of gauge unfixing to fix the gauge of the covariant equations (2.62) to just the set of equations

$$(K - \bar{K})T = 0 \tag{2.85}$$

$$dT = \bar{d}T = 0, \tag{2.86}$$

one can show that there are no propagating fields in the zero ghost number sector. However, it is not clear yet if this can be done. We have shown that the gauge covariant equations of motion in the dual sector can be obtained by successively unfixing the gauge in the above set of equations.

To summarize, the free Lagrangian for the bosonic closed string involves the coupling of the physical field to a ghost number zero field. The role of the fields in the ghost number zero sector needs to be clarified. Of particular importance is the issue of whether there are propagating fields in the ghost number zero sector, especially propagating fields that are distinct from the propagating modes of the physical field ϕ_0^0. If such fields do exist, they would raise further questions such as the boundedness of the kinetic terms in the lagrangian. Even if there are no such fields at the free level, it is not clear if this state of affairs would continue at the interacting level.* In any case, it is likely that the extra fields involved in the free theory will play a role in building a satisfactory interacting closed string field theory.

* The author would like to thank Prof.C.B. Thorn for discussions regarding these issues.

CHAPTER 3
THE BOSONIC STRING

3.1 Review of the Covariant Formalism

A string can be viewed as a one-dimensional object evolving in time. [32,16] As it does so, its shape may change and it may undergo interactions. For definiteness, we shall consider only open strings unless otherwise indicated. The points on the string can be labelled by a continuous parameter which we shall call σ. We shall choose this parameter to take the values 0 and π at the endpoints of the string; thus, in different Lorentz frames, the string would be viewed as different spacelike surfaces. It is natural to expect that the physics of the theory be independent of the choice of parametrization of the string. We shall use this as our guiding principle throughout, so that the reparametrization group is the fundamental symmetry group.[16]

The points on the string have definite spacetime coordinates $x^\mu(\sigma)$, where μ takes values from 0 to $d-1$. A natural requirement on the functions $x^\mu(\sigma)$ is that

$$x'^\mu(\sigma) = 0 \tag{3.1}$$

at the endpoints. Here and henceforth, a prime will indicate differentiation with respect to σ. These functions can therefore be expanded in terms of orthonormal even functions over the interval $[0,\pi]$. The cosines are such a set of functions; therefore we can write

$$x^\mu(\sigma) = \sqrt{2}\sum_{n=0}^{\infty} x_n^\mu \cos n\sigma. \tag{3.2}$$

Now consider making a change in σ to a new parametrization $\overline{\sigma}$, such that

$$\overline{\sigma} = \sigma + \epsilon f(\sigma) \tag{3.3}$$

We consider only changes in parametrization which leave the endpoints fixed, so that

$$f(0) = f(\pi) = 0. \tag{3.4}$$

Under such a change, which is merely a relabelling of points on the string, the spacetime coordinates must not change; we have not moved the string or changed our frame of reference. Therefore, if the $x^\mu(\sigma)$ change to new functions $\overline{x}^\mu(\overline{\sigma})$, we must have

$$\overline{x}^\mu(\overline{\sigma}) = x^\mu(\sigma) \tag{3.5}$$

or

$$\overline{x}^\mu(\sigma + \epsilon f) = x^\mu(\sigma) \tag{3.6}$$

or

$$\overline{x}^\mu(\sigma) + \epsilon f x'^\mu(\sigma) = x^\mu \tag{3.7}$$

which is correct to order ϵ^2. So the functional change in $x^\mu(\sigma)$ is

$$\delta_f x(\sigma) = \overline{x}(\sigma) - x(\sigma) = -\epsilon f x'(\sigma). \tag{3.8}$$

It is easy to check that the functional changes δ_f satisfy the infinite dimensional Lie algebra

$$[\delta_f, \delta_g] = \delta_{fg'-f'g} \tag{3.9}$$

Let us now define a functional derivative operator $\frac{\delta}{\delta x^\mu(\sigma)}$ which obeys

$$[\frac{\delta}{\delta x^\mu(\sigma_1)}, x^\nu(\sigma_2)] = \delta^\nu_\mu \delta(\sigma_1 - \sigma_2) \tag{3.10}$$

where $\delta(\sigma_1 - \sigma_2)$ is the even delta function over the interval $[0, \pi]$. The functional change in $x(\sigma)$ can be represented conveniently in terms of a generator M_f:

$$\delta_f x^\mu(\sigma) = -i\epsilon[M_f, x^\mu(\sigma)] \tag{3.11}$$

where

$$M_f = -i \int_0^\pi \frac{d\sigma}{\pi} f(\sigma) x'(\sigma) \cdot \frac{\delta}{\delta x(\sigma)}. \tag{3.12}$$

The hermitian operators M_f are then the generators of the reparametrization group, and they satisfy the algebra

$$[M_f, M_g] = iM_{fg'-f'g} \tag{3.13}$$

In order to construct a string field theory, one now considers functionals of $x(\sigma)$, namely objects like $A[x^\mu(\sigma)]$. Associated with a string $x^\mu(\sigma)$ is such a field functional $\Phi[x^\mu(\sigma)]$. This functional changes under reparametrizations as

$$\delta_f \Phi[x(\sigma)] = -i\epsilon M_f \Phi. \tag{3.14}$$

Physically, one expects the field functional to be immune to changes in the parametrization of the string:

$$M_f \Phi[x] = 0 \tag{3.15}$$

for a physical string field. We note that the generators M_f are dimensionless Lorentz scalars. They are independent of the spacetime metric and contain no time derivatives, so that they are purely kinematical objects. Upon quantization of the coordinates, however, it is not possible to implement this as a requirement on the string field, as we shall see later; it can only be implemented as a 'weak' condition, i.e., as a statement about matrix elements between physical states.

Having imposed reparametrization invariance as a fundamental kinematical constraint on string fields, the Marshall-Ramond formalism introduces invariants and covariants of the reparametrization algebra. The physical length of the string can be defined as

$$l = \int_0^\pi d\sigma \sqrt{x'^2(\sigma)}. \tag{3.16}$$

This is clearly an invariant quantity under reparametrizations, since

$$\delta_f \sqrt{x'^2(\sigma)} = \frac{2x' \cdot \delta_f x'}{2\sqrt{x'^2(\sigma)}} = -\frac{x' \cdot (fx')'}{\sqrt{x'^2(\sigma)}} = -\frac{d}{d\sigma}(f\sqrt{x'^2(\sigma)}) \tag{3.17}$$

is a total derivative and f and x'^μ vanish at the endpoints of the string. Next, under a reparametrization $\sigma \to \overline{\sigma}$, the delta function $\delta(\sigma - \sigma')$ changes to

$$\delta(\overline{\sigma} - \overline{\sigma}') = \delta(\sigma - \sigma')\frac{d\sigma}{d\overline{\sigma}}. \tag{3.18}$$

Since $x'(\sigma)$ changes to

$$\overline{x}'(\overline{\sigma}) = x'(\sigma)\frac{d\sigma}{d\overline{\sigma}} \tag{3.19}$$

this means that the quantity $\frac{\delta(\sigma-\sigma')}{\sqrt{x'^2(\sigma)}}$ is an invariant delta functional. We can therefore use $\frac{1}{\sqrt{x'^2(\sigma)}}\frac{\delta}{\delta x^\mu(\sigma)}$ as a derivative operator which is a reparametrization scalar, so that the operator

$$\tilde{M}_f = -i\int_0^\pi \frac{d\sigma}{\pi}\frac{f(\sigma)}{\sqrt{x'^2(\sigma)}}x'(\sigma) \cdot \frac{\delta}{\delta x(\sigma)} \tag{3.20}$$

is a formally invariant quantity. Similarly, the object $\frac{x'^\mu}{\sqrt{x'^2(\sigma)}}$ transforms like a scalar under reparametrizations.

One can write an action for string field theory, just as for point particles, in the form

$$S = \int_{\Sigma_i}^{\Sigma_f} \mathcal{D}x(\sigma)\mathcal{L}\left(\Phi, \frac{\delta\Phi}{\delta x}, x'(\sigma)\right). \tag{3.21}$$

Here Σ_i and Σ_f are the initial and final space-like surfaces corresponding to the locations of the string and $\mathcal{D}x(\sigma)$ is a suitable functional measure. One can then write down the Feynman path integral with a suitable measure $\mathcal{D}\Phi[x]$ if one wants to calculate amplitudes. A fundamental requirement on the Lagrangian density $\mathcal{L}$ is that it must be reparametrization invariant. It is natural to expect that, for the bosonic string, the action would yield equations of motion similar to the Klein-Gordon equation. If the equations of motion are of the form

$$A_h \Phi = 0 \tag{3.22}$$

where A_h is a 'kinetic' operator, they must be covariant(form-invariant) under reparametrizations. This means that the commutator of the reparametrization generators M_f with the A_h must itself be another A operator. Further, consistency demands that the commutator of two A operators be at most a linear combination of M and A operators. One can try to add terms to the dynamical operator $\int_0^\pi \frac{d\sigma}{\pi} \frac{\delta^2}{\delta x^2(\sigma)}$ in order to satisfy these closure properties. Further requiring that the covariant d'Alembertian be a Lorentz scalar, it is easy to see that the unique choice for the d'Alembertian is

$$\Box_g = \frac{1}{2} \int_0^\pi \frac{d\sigma}{\pi} \left[\frac{\delta^2}{\delta x^2(\sigma)} - \frac{x'^2(\sigma)}{\alpha'^2}\right] \tag{3.23}$$

where α' is a constant of dimension $(length)^2$. We will henceforth set $\alpha' = 1$. The commutation relations satisfied by $\Box$ are

$$[M_f, \Box_h] = i\Box_{fh'-f'h} \tag{3.24a}$$

$$[\Box_h, \Box_g] = iM_{hg'-h'g} \tag{3.24b.}$$

Since $\frac{1}{\sqrt{x'^2(\sigma)}} \frac{\delta}{\delta x}$ and $\frac{x'^\mu}{\sqrt{x'^2(\sigma)}}$ are reparametrization scalars, the object

$$\int_0^\pi \frac{d\sigma}{\pi} \frac{1}{\sqrt{x'^2(\sigma)}} \left(\frac{\delta^2}{\delta x^2} - x'^2\right) \tag{3.25}$$

is an invariant quantity. One can therefore write a formally reparametrization invariant Lagrangian density in the form

$$\mathcal{L} = \int_0^\pi \frac{d\sigma}{\pi}\Phi[x]\frac{1}{\sqrt{x'^2(\sigma)}}\left(-ix'\cdot\frac{\delta}{\delta x} + k(\frac{\delta^2}{\delta x^2} - x'^2)\right)\Phi[x] \qquad (3.26)$$

where k is a constant. This Lagrangian yields classical equations of motion linear in the generators $\Box$ and M. String fields satisfying the equations

$$M_f\Phi = 0 \qquad (3.27a)$$

$$\Box_h\Phi = 0 \qquad (3.27b)$$

are particular solutions of the equations of motion. We note that these are free field equations. We will shortly see that these equations only hold in the 'weak' sense once the theory is quantized.

3.2 Representations of the Reparametrization Algebra

We have seen that under a reparametrization $\sigma \to \overline{\sigma} = \sigma + \epsilon f$, the string coordinates transform like scalars:

$$\overline{x}^\mu(\overline{\sigma}) = x^\mu(\sigma). \qquad (3.28)$$

This transformation law can be generalized naturally as follows: a quantity $A(\sigma)$ is said to transform covariantly with weight w_A if under reparametrizations it satisfies[33,34,35]

$$\overline{A}(\overline{\sigma}) = A(\sigma)\left(\frac{d\sigma}{d\overline{\sigma}}\right)^{w_A}. \qquad (3.29)$$

In terms of functional changes, this means that(dropping the infinitesimal parameter ϵ)

$$\delta_f A = -(fA' + w_A f'A) \qquad (3.30)$$

The integral of any quantity which transforms with weight one is of course a reparametrization invariant, as we saw for the length of the string. We note that if $A(\sigma)$ is a covariant field, its derivative is not necessarily covariant:

$$\begin{aligned}\delta_f A' &= -(fA' + w_A f' A)' \\ &= -\left(f(A')' + (w_A+1)f'A' + w_A f'' A\right)\end{aligned} \tag{3.31}$$

Thus, $A'(\sigma)$ is covariant only if $w_A = 0$.

Given two fields A and B, what are the covariant quantities that one can form from these fields? It is clear that if the weights of A and B arew_A and w_B respectively, the product AB classically transforms covariantly with weight $(w_A + w_B)$. When the fields are quantized, however, one has to deal with operators, which could lead to ordering problems.It is easy to see that the combination $(w_A AB' - w_B A'B)$ transforms covariantly with weight $(w_A + w_B + 1)$ since the f'' terms in the transformations of A and B cancel. Upon taking more derivatives, one gets terms anomalous in derivatives of f as well as derivatives of the fields, so that it is no longer possible to form covariant combinations. Thus, one has the decomposition rule[36]

$$\mathbf{w_A} \otimes \mathbf{w_B} = (\mathbf{w_A} + \mathbf{w_B}) \oplus (\mathbf{w_A} + \mathbf{w_B} + 1). \tag{3.32}$$

The transformation rule for $A(\sigma)$ can be written conveniently in terms of the generator

$$M_f = -i\int_0^\pi \frac{d\sigma}{\pi}(fA' + w_A f' A)\frac{\delta}{\delta A(\sigma)} \tag{3.33}$$

as

$$\delta_f A = -i[M_f, A]. \tag{3.34}$$

Next we turn to representations in terms of non-covariant fields, or gauge representations.[36] Consider an operator $\mathcal{O}$ which is defined to act on fields

of weight w and produce fields of weight $w + \Delta$. Such an operator has the transformation law

$$-\delta_f \mathcal{O} = f\mathcal{O}' + \Delta f'\mathcal{O} - [\mathcal{O}, f]\frac{d}{d\sigma} - w[\mathcal{O}, f']. \tag{3.35}$$

As a particular example of this, consider the operator

$$P = E_1 \frac{d}{d\sigma} + E_2 \tag{3.36}$$

which acts on a covariant field A to give a field with weight $(w_A + \Delta)$. Then from

$$\begin{aligned} \delta_f(PA) =& \delta_f(E_1 A' + E_2 A) \\ =& (\delta_f E_1)A' + E_1 \delta_f A' + (\delta_f E_2)A + E_2 \delta_f A \\ =& -\left(f(PA)' + (w_A + \Delta) f' PA\right) \end{aligned} \tag{3.37}$$

we can read off the transformations of E_1 and E_2:

$$\delta_f E_1 = -\left(fE_1' + (\Delta - 1) f' E_1\right) \tag{3.38}$$

$$\delta_f E_2 = -\left(fE_2' + \Delta f' E_2 - w f'' E_1\right). \tag{3.39}$$

We note that E_1 transforms covariantly, unlike E_2. However, the combination

$$E = wE_1' + (\Delta - 1)E_2 \tag{3.40}$$

is a new covariant field provided $\Delta \neq 1$, so that the representation is reducible in this case. One can form a covariant derivative which raises the weight of a field by one by taking $E_1 = 1$; then

$$PA = (\frac{d}{d\sigma} + \hat{w}C)A \tag{3.41}$$

has weight $(w_A + 1)$ if C transforms as

$$\begin{aligned} \delta_f C &= -(fC' + f'C) + f'' \\ &= -(fC)' + f''. \end{aligned} \tag{3.42}$$

Here $\hat{w}$ is the weight operator; its value is simply the weight of the field on which it acts. The fact that C transforms inhomogeneously makes it similar to a gauge field or a connection.

One can form from the field C the quantities $e(\sigma)$ and $\phi(\sigma)$ defined by

$$C(\sigma) = -\frac{1}{w}\phi' \tag{3.43}$$

and

$$e(\sigma) = \exp(\phi(\sigma)). \tag{3.44}$$

Here w is a scale factor which is simply the classical weight of the covariant field $e(\sigma)$. The field ϕ, being the logarithm(at least classically)[33] of e, transforms inhomogeneously:

$$\delta_f \phi = -(f\phi' + wf'). \tag{3.45}$$

We shall see later that ordering effects actually change the classical weight of the field $e(\sigma)$. We note that the covariant derivative of e is zero, which is analogous to the statement in Riemannian geometry that the metric is covariantly constant. The 'einbein' field $e(\sigma)$ can therefore be thought of as a metric in the space of one-dimensional reparametrizations.[36]

As we have seen, reparametrizations may be conveniently described in terms of generators involving functional derivatives. Classically, the functional derivative $\frac{\delta}{\delta e(\sigma)}$ has weight $(1-w)$, since $e(\sigma)$ has weight w. Therefore, in general, $\frac{\delta^2}{\delta e^2(\sigma)}$ does not transform in the same manner as $\frac{\delta^2}{\delta x^2(\sigma)}$ does, i.e., with weight two. As we shall shortly see, we would like to construct a dynamical operator from the field $e(\sigma)$. Since $\frac{\delta}{\delta\phi}$ has weight one, it turns out to be more convenient to work with $\phi(\sigma)$ rather than $e(\sigma)$.

The exponentials $e^{a\phi}$ classically transform covariantly with weight aw. What polynomial covariants can one form from $\phi(\sigma)$? Since $\phi(\sigma)$ transforms

inhomogeneously, the answer is actually none. The closest one can get to a covariant quantity is the combination $(\phi^2 - 2w\phi'')$. This transforms anomalously:

$$\delta_f(\phi^2 - 2w\phi'') = -f(\phi^2 - 2w\phi'')' - 2f'(\phi^2 - 2w\phi'') + 2w^2 f''. \qquad (3.46)$$

This combination then is the analogue of x'^2, so that we can use the object

$$\frac{\delta^2}{\delta\phi^2(\sigma)} - (\phi^2 - 2w\phi'') \qquad (3.47)$$

as a dynamical operator(upto a constant) for the field $\phi(\sigma)$.

3.3 Quantization and Construction of a Dynamical Invariant

The functional derivatives $\frac{\delta}{\delta x^\mu(\sigma)}$ and the coordinates $x^\mu(\sigma)$ can be expanded in Fourier modes as

$$x^\mu(\sigma) = x_0^\mu + \sqrt{2}\sum_{n=1}^{\infty} x_n^\mu \cos n\sigma \qquad (3.48)$$

$$\frac{\delta}{\delta x^\mu(\sigma)} = \frac{\partial}{\partial x_0^\mu} + \sqrt{2}\sum_{n=1}^{\infty} \frac{\partial}{\partial x_n^\mu} \cos n\sigma \qquad (3.49)$$

with

$$[\frac{\partial}{\partial x_n^\mu}, x_m^\nu] = \delta_\mu^\nu \delta_{n,m}. \qquad (3.50)$$

To quantize the string, we now introduce harmonic oscillator modes α_n^μ defined by

$$x_n^\mu = \frac{i}{n\sqrt{2}}(\alpha_n^\mu - \alpha_{-n}^\mu) \qquad (3.51)$$

$$\frac{\partial}{\partial x_n^\mu} = \frac{i}{\sqrt{2}}(\alpha_{n,\mu} + \alpha_{-n,\mu}) \qquad (3.52)$$

for $n \neq 0$. These satisfy

$$[\alpha_m^\mu, \alpha_n^\nu] = g^{\mu\nu} m \delta_{m+n,0}. \qquad (3.53)$$

The zero mode $\alpha_{0\mu} = -i\frac{\partial}{\partial x_0^\mu}$ satisfies

$$[x_0^\mu, \alpha_0^\nu] = ig^{\mu\nu}. \tag{3.54}$$

The vacuum state is defined by

$$\alpha_m^\mu |0>= 0 \tag{3.55}$$

for all $m > 0$. The generators

$$M_f = -i\int_0^\pi \frac{d\sigma}{\pi} f x' \cdot \frac{\delta}{\delta x} \tag{3.56}$$

can be expanded in a Fourier series in terms of sine functions. We note that once we introduce harmonic oscillators, these generators, which are formally metric independent, can be written in terms of the Minkowski metric. This simply corresponds to the fact that we have broken general covariance by introducing harmonic oscillators in flat space. The M_f's can also be written in the form

$$M_f = \frac{1}{4}\int_0^\pi \frac{d\sigma}{\pi} f \left({x'_R}^2 - {x'_L}^2\right) \tag{3.57}$$

with

$${x'_L}^\mu(\sigma) = x'^\mu(\sigma) + i\frac{\delta}{\delta x_\mu(\sigma)} \tag{3.58}$$

and

$${x'_R}^\mu(\sigma) = x'^\mu(\sigma) - i\frac{\delta}{\delta x_\mu(\sigma)} \tag{3.59}$$

The combinations $x_L(\sigma)$ and $x_R(\sigma)$ are expanded in terms of exponentials:

$$x_L^\mu(\sigma) = x_0^\mu - \alpha_0^\mu \sigma + i\sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{in\sigma} \tag{3.60}$$

$$x_R^\mu(\sigma) = x_0^\mu + \alpha_0^\mu \sigma + i\sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\sigma} \tag{3.61}$$

They are related by parity: $x_L(-\sigma) = x_R(\sigma)$. It is more convenient to work with exponential functions now that we have split the coordinates in the above manner. We therefore extend the range of σ to cover $[-\pi, \pi]$. Then the operator

$$M(\sigma) = -ix' \cdot \frac{\delta}{\delta x} \tag{3.62}$$

has Fourier modes

$$\begin{aligned} M_n = M_{e^{in\sigma}} &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} e^{in\sigma} \left(M^L(\sigma) + M^R(\sigma)\right) \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} e^{in\sigma} ({x'_R}^2 - {x'_L}^2) \\ &= L_n - L_{-n} \end{aligned} \tag{3.63}$$

where the L's are the Virasoro operators

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m \tag{3.64}$$

We cannot demand

$$M_n|\psi >= (L_n - L_{-n})|\psi >= 0 \tag{3.65}$$

as a physical state condition since we have already chosen our vacuum to be annihilated by the positive modes α_m $(m > 0)$. We can at best impose this as a condition on matrix elements of physical states. The normal-ordered Virasoro operators satisfy the anomalous algebra[37]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{d}{12}(n^3 - n)\delta_{n+m,0} \tag{3.66}$$

The modes M_n of $M(\sigma)$ satisfy the anomaly-free algebra

$$[M_n, M_m] = (n-m)M_{n+m} - (n+m)M_{n-m}. \tag{3.67}$$

This means that the reparametrization generators are covariant operators even upon normal-ordering. We note that, by construction, $M(\sigma)$(as well as its left and right-moving pieces) is a weight-two operator.

The normal-ordered exponentials $e^{ik\cdot x_L(\sigma)}$ transform covariantly[38,39] with weight $k^2/2$. Similar normal-ordered exponentials with the coordinates $x^\mu(\sigma)$ are not covariant since they are afflicted with ordering anomalies. Polynomials of order greater than two in $x^\mu(\sigma)$ and its derivatives are not covariant since they contain operator-valued anomalies under reparametrizations.

We now turn to the dynamical operator

$$\Box_h = \frac{1}{2}\int_0^\pi \frac{d\sigma}{\pi} h(\sigma)\left(\frac{\delta^2}{\delta x^2(\sigma)} - x'^2(\sigma)\right). \tag{3.68}$$

This can be rewritten in terms of x'_L and x'_R as

$$\Box_h = -\frac{1}{4}\int_0^\pi \frac{d\sigma}{\pi} h(\sigma)\left((x'^2_L(\sigma) + x'^2_R(\sigma)\right) \tag{3.69}$$

The density $\Box(\sigma)$ has Fourier modes

$$\Box_n = \Box_{e^{in\sigma}} = -(L_n + L_{-n}) \tag{3.70}$$

Classically, the $\Box$ operator transforms covariantly with weight two, as can be seen from its commutations with M_f. The normal-ordered operator, however, transforms anomalously due to the central charge term in the Virasoro algebra:

$$[M_n, \Box_m] = (n-m)\Box_{n+m} + (n+m)\Box_{n-m} - \frac{D}{6}(n^3 - n)(\delta_{n+m,0} + \delta_{n-m,0}). \tag{3.71}$$

So we cannot use the $\Box$ operator as a covariant equation of motion, unless we can somehow get rid of the anomaly. Also, we cannot yet construct a measure of suitable weight for use in the construction of an invariant operator(as in (3.26)); objects like $\sqrt{x'^2(\sigma)}$ are now ill-defined since we have quantized the theory.

As a possible solution to both of these problems, we introduce an extra 'ein-bein' field variable[40,41,33] $e(\sigma)$ (the same one which appeared in the covariant

derivative in the space of one-dimensional reparametrizations) and quantize it. The motivation for introducing the einbein field comes from the analogy with the case of the point particle. The free point particle action (with $\dot{x} \equiv \frac{dx}{d\tau}$)

$$m \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{x}^2} \tag{3.72}$$

can be replaced by[42,43,44]

$$\int d\tau \{[1/e(\tau)]\dot{x}^2 + m^2 e(\tau)\} \tag{3.73}$$

where $e(\tau)$ is an einbein field which transforms as a total derivative under reparametrizations in τ. So the einbein field serves in this case to eliminate the need for square roots, and at the same time provides an action for the massless point particle. The einbein field in our case is also introduced with the view of eliminating square roots in the action; it acts as a 'metric' in the space of one-dimensional reparametrizations. The price we pay is just that the string field now also depends on the extra field we have introduced.

It turns out to be more convenient to work with the field $\phi = \ln e(\sigma)$ rather than with e itself.[33] Now the string functional Φ also depends on $\phi(\sigma)$, in addition to the coordinates.[19] This field has the inhomogeneous transformation law mentioned in the previous section:

$$\delta_f \phi = -(f\phi' + wf'). \tag{3.74}$$

We could work directly with the covariant field $e(\sigma)$, but then we would run into problems when we tried to construct a dynamical operator, since the functional derivative would then have a weight different from one. The reparametrization generator for this field takes the form[35]

$$M_f^{\phi} = -i \int_0^{\pi} \frac{d\sigma}{\pi} (f\phi' + wf') \frac{\delta}{\delta\phi} \tag{3.75}$$

We can expand $\phi(\sigma)$ in modes, just like the coordinates:

$$\phi(\sigma) = \phi_0 + \sqrt{2}\sum_{n=1}^{\infty} \phi_n \cos n\sigma \tag{3.76a}$$

$$\frac{\delta}{\delta\phi} = \frac{\partial}{\partial\phi_0} + \sqrt{2}\sum_{n=1}^{\infty} \frac{\partial}{\partial\phi_n}\cos n\sigma \tag{3.76b}$$

We can quantize ϕ just as we did the coordinates by introducing harmonic oscillators:

$$\phi_n = \frac{i}{n\sqrt{2}}(\beta_n - \beta_{-n}) \tag{3.76c}$$

$$\frac{\partial}{\partial\phi_n} = \frac{i\eta}{2}(\beta_n + \beta_{-n}) \tag{3.76d}$$

$$\phi(\sigma) = \frac{\phi_L(\sigma) + \phi_R(\sigma)}{2} \tag{3.76e}$$

$$\phi_L(\sigma) = \phi_0 - \beta_o\sigma + i\sum_{n\neq 0} \frac{\beta_n}{n} e^{in\sigma} \tag{3.76f}$$

$$\phi_R(\sigma) = \phi_0 + \beta_o\sigma + i\sum_{n\neq 0} \frac{\beta_n}{n} e^{-in\sigma} \tag{3.76g}$$

$$\phi_L(-\sigma) = \phi_R(\sigma) \tag{3.76h}$$

The modes β_n satisfy

$$[\beta_n, \beta_m] = \eta n \delta_{n+m,0} \tag{3.77a}$$

$$[\phi_0, \beta_0] = i\eta \tag{3.77b}$$

Here the parameter η takes the values ± 1;$\eta = -1$ means that $\phi(\sigma)$ has ghost-like excitations. The vacuum state is defined by

$$\beta_n |0>= 0 \tag{3.78}$$

for all $n > 0$. The left and right-moving pieces of ϕ can be written as

$$\phi_L' = \phi' + i\eta\frac{\delta}{\delta\phi} \tag{3.79}$$

$$\phi'_R = \phi' - i\eta\frac{\delta}{\delta\phi} \tag{3.80}$$

Correspondingly, the dynamical operator is

$$\Box_f^\phi = \frac{\eta}{2}\int_0^\pi \frac{d\sigma}{\pi}\left(\frac{\delta^2}{\delta\phi^2} - (\phi'^2 - 2w\phi'')\right) \tag{3.81}$$

as mentioned in the previous section. We note that $\eta = -1$ corresponds to negative kinetic energy for the field $\phi(\sigma)$. The factor of η here is necessary for separability of the left and the right-moving pieces of the M and $\Box$ operators. We can write

$$\begin{aligned} M_f &= M_f^L + M_f^R \\ &= -\frac{\eta}{2}\int_0^\pi \frac{d\sigma}{\pi}\left(f\frac{{\phi'_L}^2 - {\phi'_R}^2}{2} - wf(\phi''_L - \phi''_R)\right) \end{aligned} \tag{3.82}$$

and

$$\begin{aligned} \Box_f &= M_f^L - M_f^R \\ &= -\frac{\eta}{2}\int_0^\pi \frac{d\sigma}{\pi}\left(f\frac{{\phi'_L}^2 + {\phi'_R}^2}{2} - wf(\phi''_L + \phi''_R)\right) \end{aligned} \tag{3.83}$$

or, in terms of Fourier modes,

$$M_n^\phi = L_n^\phi - L_{-n}^\phi \tag{3.84}$$

$$\Box_n = -(L_n^\phi + L_{-n}^\phi). \tag{3.85}$$

Here the Virasoro operators for the field ϕ are

$$L_n^\phi = \eta\left(\frac{1}{2}\sum_m(\beta_{n-m}\beta_m) + \frac{iwn}{2}\beta_n\right) \tag{3.86}$$

The normal-ordered L_n^ϕ obey the algebra

$$[L_n^\phi, L_m^\phi] = (n-m)L_{n+m}^\phi + \frac{1}{12}(12w^2n^3\eta + n^3 - n)\delta_{n+m,0} \tag{3.87}$$

The linear term in the anomaly can be absorbed by a shift in L_0; the cubic term in the anomaly of the algebra of the total Virasoro generators $(L_n^\phi + L_n^x)$ vanishes for

$$d + 12w^2\eta + 1 = 0. \tag{3.88}$$

Clearly, η must be minus one to yield sensible values of D(since w is real).

The only normal-ordered covariants one can form from ϕ are the normal-ordered exponentials $e^{a\phi_L(\sigma)}$ (and similarly for $\phi_R(\sigma)$) defined by[38,39]

$$: e^{a\phi_L(\sigma)} := \exp(ai\sum_{n<0}\frac{\beta_n}{n}e^{in\sigma})e^{a\phi_0}e^{-a\sigma(\beta_0+ia/2)}\exp(ai\sum_{n>0}\frac{\beta_n}{n}e^{in\sigma}). \tag{3.89}$$

The quantity $e^{a\phi_L(\sigma)}$ transforms covariantly with the weight $a(w - a\eta/2)$.

We need a weight minus-one object as an integration measure $\mu(\sigma)$ in order to construct a dynamical invariant operator of the form (as in (3.26))

$$K = \int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\mu(\sigma)M^L(\sigma) \tag{3.90}$$

where $M^L(\sigma)$ is the total left-moving part of the reparametrization density including coordinate and ghost contributions. We note that we cannot mix left and right-moving modes here since it would lead to equations of motion that are inconsistent with the definition of the vacuum state. Since the exponential $e^{a\phi_L}$ is the only possible covariant that could provide us with a suitable measure, we must have

$$a(w - a\eta/2) = -1. \tag{3.91}$$

This gives us

$$w = -\frac{a}{2} - \frac{1}{a}. \tag{3.92}$$

Requiring the exponential $e^{a\phi_L}$ to be single-valued as σ changes from $-\pi$ to π, we see that a must be an integer since the eigenvalues of β_0 increase in steps of

i. We note that β_0, being the 'momentum' of the ϕ field, is a reparametrization invariant. The states of the theory are thus labelled by their eigenvalues under β_0, in addition to the values of the spacetime momentum. These eigenvalues label the ghost numbers of the states. Since d must be a positive integer, we see from (3.88) that w must be a half-integer. So we can only have $a = \pm 1$ or $a = \pm 2$; we have $w = 3/2$ for $a = -1$ or -2 and $w = -3/2$ for $a = +1$ or $+2$. For either of these possibilities, the theory predicts twenty-six spacetime dimensions.

The operator K must be an overall normal-ordered expression for it to make sense. This means that we still have to check the invariance of K after it has been normal-ordered. Let us set

$$Q = \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} : e^{a\phi_L} M^L(\sigma) : \tag{3.93}$$

and check its invariance. We find[35]

$$[L_n, Q] = \frac{n(n+1)}{2} \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} e^{-in\sigma} (i\frac{d}{d\sigma} - \frac{a^2}{2} - \frac{aw}{3} - \frac{2}{3} naw) e^{a\phi_L}. \tag{3.94}$$

We see that the right-hand side is a total derivative if and only if $a^2 = 1$ and $aw = -3/2$. We make the choice $a = 1$, corresponding to $w = -3/2$. So we get a unique invariant scalar operator in twenty-six dimensions. This is of course the usual BRST charge, and it is not hard to check that it is nilpotent. The field $\phi(\sigma)$ is then the bosonized ghost field. From the operator product

$$e^{a\phi_L(\sigma_1)} e^{b\phi_L(\sigma_2)} = e^{a\phi_L(\sigma_1) + b\phi_L(\sigma_2)} \left(-2i\sin\frac{\sigma_1 - \sigma_2}{2} \right)^{-\eta ab} \tag{3.95}$$

we can see that $c_L \equiv e^{\phi_L}$ and $b_L \equiv e^{-\phi_L}$ are conjugate anti-commuting fields; these are the usual anti-commuting ghost and anti-ghost respectively of the bosonic string.

As mentioned earlier, the states are labelled by the eigenvalues of $-i\beta_0$, which are half-integer; this is simply the ghost-number operator. The free field theory action is given by

$$S =< \Phi|Q|\Phi > \tag{3.96}$$

and it yields the usual equation of motion

$$Q|\Phi >= 0. \tag{3.97}$$

As a consequence of the nilpotency of Q, this has the well-known gauge invariance $|\Phi >\rightarrow |\Phi > +Q|\Lambda >$ which eliminates states of negative norm.[45] In the next chapter we shall generalize the theory to include fermions.

CHAPTER 4
SUPER-REPARAMETRIZATIONS

4.1 The Covariant Formalism

The Marshall-Ramond extension to include super-reparametrizations in the formalism is based on the introduction of two anti-commuting quantities, the generalized Dirac gamma matrices $\Gamma_i^\mu(\sigma)$ $(i = 1, 2)$. These hermitian operators obey the anti-commutation rules[16]

$$\{\Gamma_i^\mu(\sigma), \Gamma_j^\nu(\sigma')\} = 2\delta_{ij} g^{\mu\nu} \delta(\sigma - \sigma'). \tag{4.1}$$

As for the coordinates, the delta function here is defined over the interval $[0,\pi]$. Since the delta function is a weight-one object, these fermions are weight-one-half objects under reparametrizations. Two sets of matrices are necessary for the construction of a dynamical operator, as will be seen later.

We note that these matrices can be replaced by the equivalent set $\Gamma^\mu(\sigma)$, $\frac{\delta}{\delta\Gamma^\mu(\sigma)}$, defined by

$$\Gamma_1^\mu(\sigma) = \Gamma^\mu(\sigma) + \frac{\delta}{\delta\Gamma_\mu(\sigma)} \tag{4.2}$$

$$\Gamma_2^\mu = i\left(\frac{\delta}{\delta\Gamma_\mu(\sigma)} - \Gamma(\sigma)\right). \tag{4.3}$$

Since these are weight-one-half quantities, the reparametrization generators are given by

$$M_f = -i\int_0^\pi \frac{d\sigma}{\pi}(f\Gamma' + \frac{1}{2}f'\Gamma)\cdot\frac{\delta}{\delta\Gamma} \tag{4.4}$$

or equivalently by

$$M_f = -\frac{i}{4}\int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\Gamma_i' \cdot \Gamma_i. \tag{4.5}$$

This generator acts on string functionals which are spacetime spinors. In particular, the wave functional of the string is such a spinor. In order to obtain Dirac-like first-order equations of motion for the string field, the Lagrangian density must also contain a first-order operator.

Since $\Gamma_i^\mu(\sigma)$ is a weight-one-half quantity, the objects $(x'^2)^{-1/4}\Gamma_i^\mu(\sigma)$ transform as reparametrization scalars. Using this fact, one can build a Lagrangian density of the form

$$\mathcal{L} = \int_0^\pi \frac{d\sigma}{\pi}(x'^2)^{-1/4}\left(i\Gamma_1 \cdot \frac{\delta}{\delta x} + \Gamma_2 \cdot x'\right) \tag{4.6}$$

where the i is included for hermiticity. This Lagrangian density is reparametrization-invariant by construction. Used in an action of the form

$$S = < \Psi|\mathcal{L}|\Psi > \tag{4.7}$$

it yields equations of motion of the form

$$\int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left(i\Gamma_1 \cdot \frac{\delta}{\delta x} + \Gamma_2 \cdot x'\right)|\Psi >= 0. \tag{4.8}$$

The operators

$$\not{D}_f \equiv \int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left(i\Gamma_1 \cdot \frac{\delta}{\delta x} + \Gamma_2 \cdot x'\right) \tag{4.9}$$

satisfy the classical algebra

$$[M_f, \not{D}_g] = i\not{D}_{fg'-f'g/2} \tag{4.10a}$$

$$\{\not{D}_f, \not{D}_g\} = -4\Box_{fg} \tag{4.10b}$$

$$[\not{D}_f, \Box_g] = i\not{Q}_{fg'/2-f'g}. \tag{4.10c}$$

Here M_f stands for the total reparametrization generators including the coordinates and the Γ's. The operator $\not{Q}_f$ is defined by interchanging Γ_1 and Γ_2 in $\not{D}_f$:

$$\not{Q}_f \equiv \int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left(i\Gamma_2 \cdot \frac{\delta}{\delta x} + \Gamma_1 \cdot x'\right) \tag{4.11}$$

The $\Box_f$ operator now includes contributions from the Γ's:

$$\Box_f = \frac{1}{2}\int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left(\frac{\delta^2}{\delta x^2(\sigma)} - x'^2(\sigma) + \frac{i}{2}(\Gamma_2(\sigma)\Gamma_1'(\sigma) + \Gamma_1(\sigma)\Gamma_2'(\sigma)\right). \tag{4.12}$$

The algebra is completed by noting that the following commutations hold:

$$\{\not{Q}_f, \not{Q}_g\} = -4\Box_{fg} \tag{4.13a}$$

$$\{\not{D}_f, \not{Q}_g\} = -4M_{fg} \tag{4.13b}$$

$$[\not{Q}_f, \Box_g] = i\not{D}_{fg'/2-f'g} \tag{4.13c}$$

$$[\not{Q}_f, M_g] = i\not{Q}_{fg'/2-f'g} \tag{4.13d}$$

The above equations of motion are then covariant, at least clasically. Upon quantization, however, an anomaly arises in (4.13a) and in (4.10b), which means that the corresponding equations of motion are no longer covariant. The anomaly needs to be cancelled, and we can do this as for the bosonic string by simply adding extra fields. Before doing this, however, one needs to take a closer look at the super-reparametrization algebra and its representations, to which we now turn. The results obtained in the rest of this chapter are based on the author's work in ref.36.

4.2 Linear Representations of the Super-reparametrization Algebra

For superstring field theory we seek a kinematical supersymmetry[36] transformation $\not{\delta}_f$ which is the "square root" of the reparametrization δ_f in the sense that

$$[\not{\delta}_f(\xi_1), \not{\delta}_g(\xi_2)]F(\sigma) = -2\delta_{fg}(\xi_1\xi_2)F(\sigma) \tag{4.14}$$

for any field $F(\sigma)$, where the ξ's are anticommuting parameters. The commutation relations of the $\oint$'s with the reparametrizations δ can be determined from the Jacobi identity

$$\Big[[\oint_f, \oint_g], \delta_h\Big] + \Big[[\delta_h, \oint_f], \oint_g\Big] + \Big[[\oint_g, \delta_h], \oint_f\Big] = 0. \qquad (4.15)$$

We first note that the commutator of a reparametrization δ_h with a super-reparametrization $\oint_f$ must be bilinear in f, h and their derivatives; this is clear from (4.15) and (4.14). Furthermore, derivatives of order higher than one are excluded due to the presence of the first term in (4.15) (since this identity should hold for arbitrary functions). The commutator must therefore have the form

$$[\delta_h(\epsilon), \oint_f(\xi)] = \alpha \oint_{hf' + \beta h' f}(\epsilon\xi). \qquad (4.16)$$

Using this relation and (4.14) in (4.15), we find that $\alpha = 1$ and $\beta = -\frac{1}{2}$, *i.e.*

$$[\delta_h(\epsilon), \oint_f(\xi)] = \oint_{hf' - h'f/2}(\epsilon\xi). \qquad (4.17)$$

Henceforth, the parameters ϵ and ξ will not be indicated explicitly unless clarity warrants it.

Given fields transforming in a specified manner under reparametrizations we can deduce their possible transformation properties under super-reparametrizations. First consider the case of a field $a(\sigma)$, either commuting or anticommuting, transforming covariantly under reparametrizations with weight w_a, for which we postulate the transformation law

$$\oint_f a = -fb \qquad (4.18)$$

where b is a field of opposite type (commuting or anticommuting) from a. (4.17) tells us that

$$-f\delta_g b = (\oint_f \delta_g - \oint_{gf' - g'f/2})a \qquad (4.19)$$

and upon evaluating the right hand side of (4.19) we find

$$\delta_g b = -gb' - (w_a + \frac{1}{2})g'b \;, \tag{4.20}$$

i.e. b transforms covariantly with weight $w_b = w_a + \frac{1}{2}$. Assuming that a and b form a closed multiplet involving no other fields(we shall show later that adding extra fields does not generate new irreducible representations), the most general form for the transformation of b under a super-reparametrization is

$$\not\delta_f b = \sum_n A_n \frac{d^n a}{d\sigma^n} \tag{4.21}$$

where the A_n's are functions of f and its derivatives. Using (4.14) with $g = f$, we find

$$\not\delta_f \not\delta_f b = -\delta_{ff} b \tag{4.22}$$

so that

$$-\sum_n A_n \frac{d^n}{d\sigma^n} fb = ffb' + 2w_b ff'b. \tag{4.23}$$

Since b and its derivatives are all independent, we can equate coefficients on either side to solve for the A_n's. We find that the only non-zero A_n's are $A_0 = -2w_a f'$ and $A_1 = -f$, *i.e.*

$$\not\delta_f b = -(fa' + 2w_a f'a). \tag{4.24}$$

We have discovered one type of multiplet[36] on which the super-reparametrization algebra is represented. The representation can be written in matrix form:

$$\not\delta_f \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 0 & f \\ f\frac{d}{d\sigma} + 2w_a f' & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{4.25}$$

whereas the transformation δ_f is written as

$$\delta_f \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} f\frac{d}{d\sigma} + w_a f' & 0 \\ 0 & f\frac{d}{d\sigma} + (w_a + \frac{1}{2})f' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.26}$$

The representation is the same regardless of the Grassmann character of a or b. For this type of multiplet, we will refer to the component a transforming according to (4.18) as the *light* component, and to b which transforms according to (4.24) as the **heavy** component. An important difference between the two components is that if the integral of the heavy component is reparametrization invariant (*i.e.* if it has weight one), then it is also super-reparametrization invariant, as is evident from the transformation law (4.24). The integral over the light component is never super-reparametrization invariant.

An example of this type of representation is provided by the string coordinates x^μ. These transform according to (4.18) into the generalized Dirac matrices Γ^μ:

$$\not{\delta}_f x^\mu = -f\Gamma^\mu \tag{4.27}$$

$$\not{\delta}_f \Gamma^\mu = -f x^{\mu\prime} \tag{4.28}$$

Because x^μ has weight zero, the multiplet (Γ^μ, x'^μ) also transforms as (4.25), with Γ^μ as the light component. This multiplet is of more direct use in string field theory because it is translationally invariant.

Given two doublets (a, b) and (c, d), it will be useful to know all of the different covariant super-reparametrization representations which can be built out of products of these fields and their derivatives. One can form eight quantities which transform covariantly:

$$\begin{array}{ll}
\text{weight } w: & A_1 = ac \\
\text{weight } w+\frac{1}{2}: & A_2 = ad \ \text{ and } A_3 = bc \\
\text{weight } w+1: & A_4 = bd \ \text{ and } A_5 = w_c a'c - w_a ac' \\
\text{weight } w+\frac{3}{2}: & A_6 = (w_c+\frac{1}{2})a'd - w_a ad' \ \text{ and} \\
 & A_7 = w_c b'c - (w_a+\frac{1}{2})bc' \\
\text{weight } w+2: & A_8 = (w_c+\frac{1}{2})b'd - (w_a+\frac{1}{2})bd'
\end{array} \tag{4.29}$$

In these equations, $w \equiv w_a + w_c$. Among these quantities, three combinations may be identified as doublets:

$$(A_1, \pm A_2 + A_3), \qquad \text{with weight } (w, w+\frac{1}{2}) \qquad (4.30a)$$

$$(w_a A_2 \mp w_c A_3, \pm A_5 + w A_4), \qquad \text{with weight } (w+\frac{1}{2}, w+1) \quad (4.30b)$$

$$(\pm 2A_5 + A_4, \pm 2A_7 + 2A_6), \qquad \text{with weight } (w+1, w+\frac{3}{2}) \quad (4.30c)$$

The upper (lower) sign of the $\pm$'s in these equations is to be read in the case where a is the commuting (anticommuting) member of its multiplet. The heavy component of both (4.30b) and (4.30c) reduce to total derivatives in the cases in which their weight is one, so they yield only trivial invariants. The remaining two quantities in (4.29) are members of a multiplet containing non-covariant quantities.

We have thus demonstrated the decomposition[36]

$$\mathbf{2_w} \otimes \mathbf{2_v} = \mathbf{2_{v+w}} \oplus \mathbf{2_{v+w+\frac{1}{2}}} \oplus \mathbf{2_{v+w+1}} \oplus (\mathrm{non-covariant}) \qquad (4.31)$$

We will use the symbols $\otimes_a, \otimes_b, \otimes_c$ to denote the three ways of combining two doublets to obtain a third given in (4.30); *i.e.*

$$(a,b) \otimes_a (c,d) \equiv (ac, \pm ad + bc) \ ,$$

$$(a,b) \otimes_b (c,d) \equiv (w_a ad \mp w_c bc, \pm w_a ac' \mp w_c a'c + wbd) \ ,$$

$$(a,b) \otimes_c (c,d) \equiv (\pm 2(w_c a'c - w_a ac') + bd,$$
$$\pm 2w_c b'c \mp (2w_a+1)bc' + (2w_c+1)a'd - 2w_a ad') \ .$$

Note that for the $\otimes_b$ and $\otimes_c$ coupling schemes, the heavy component can only yield trivial invariants.

The fact that the only covariant representation of the super-reparametrization algebra found in the direct product of two doublets is again a doublet suggests that no other covariant irreducible representations exist. We shall now

prove that the doublet representation given by (4.18) and (4.24) is the only irreducible linear representation of the super-reparametrization algebra whose basis elements are a finite number of fields which transform covariantly under reparametrizations. We will show that given a set of covariant fields which transform into each other under super-reparametrizations, the representation can be reduced into a series of doublets.[36]

We will use the notation $a_{w,i}$ to denote the ith field of weight w in the collection, where $i = 1$ to N_w for each value of w. Consider the fields $a_{w_0,i}$, where w_0 is the lowest weight in the set. Since the super-reparametrizations increase the weight by $\frac{1}{2}$, these fields must transform into weight $w_0 + \frac{1}{2}$ fields. We can choose the basis for these fields so that

$$\not{\delta}_f a_{w_0,i} = -f a_{w_0+\frac{1}{2},i}, \qquad i = 1 \text{ to } N_{w_0} \tag{4.32}$$

Applying a second super-reparametrization operator, the covariance of $a_{w_0,i}$ requires

$$\not{\delta}_f a_{w_0+\frac{1}{2},i} = -(f a'_{w_0,i} + 2w_0 f' a_{w_0,i}) \qquad i = 1 \text{ to } N_{w_0} \tag{4.33}$$

i.e. the combinations $(a_{w_0,i}, a_{w_0+\frac{1}{2},i})$ form N_{w_0} independent doublets. We now show that with an appropriate choice of basis, the elements of these doublets do not appear elsewhere in the representation. First consider the other elements, i.e. $N_{w_0} < i \leq N_{w_0+\frac{1}{2}}$. The most general possible transformation law satisfying (4.17) for these elements is

$$\not{\delta}_f a_{w_0+\frac{1}{2},i} = -\sum_{j=1}^{N_{w_0}} A_{ij}(f a'_{w_0,j} + 2w_0 f' a_{w_0,j}) - f a_{w_0+1,i} \tag{4.34}$$

with an appropriate choice of basis for the weight w_0+1 elements. By changing the basis for the weight $w_0+\frac{1}{2}$ elements we can obtain elements which do not transform into the weight w_0 elements. Redefining

$$a_{w_0+\frac{1}{2},i} \to a_{w_0+\frac{1}{2},i} - \sum_{j=1}^{N_{w_0}} A_{ij} a_{w_0+\frac{1}{2},j}, \tag{4.35}$$

we obtain

$$\not{f} a_{w_0+\frac{1}{2},i} = -f a_{w_0+1,i} \tag{4.36}$$

We now show that the elements in the doublets $(a_{w_0}, a_{w_0+\frac{1}{2}})$ do not appear elsewhere in the algebra. (Here the subscripts i are left as implicitly understood). Let a_w be the first (i.e. lowest weight) element whose transformation law involves one of these elements. Then there are two cases to be considered:

1) $w - w_0 \equiv n$ is an integer, and the transformation of a_w involves $a_{w_0+\frac{1}{2}}$. a_w could possibly have the transformation law

$$\not{f}_f a_w = \sum_m A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m} a_{w_0+\frac{1}{2}}}{d\sigma^{n-m}} + X_f \tag{4.37}$$

where A_i are coefficients and X_f is some quantity which does not involve the elements in the doublet. X_f is found to transform to

$$\not{f}_f X_f = f^2 a_w' + 2w f f' a_w + \sum_m A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m}}{d\sigma^{n-m}} (f a_{w_0}' + 2w_0 f' a_{w_0}). \tag{4.38}$$

The transformation of X_f involves a_{w_0}; since we assumed that no field of lower weight than w has this property, X_f must have higher weight; the only possibility is

$$X_f = f a_{w+\frac{1}{2}}. \tag{4.39}$$

Since X_f has no derivatives of f, the only possible A_i's which could be nonzero are those which are multiplied by f, which in this case is only A_0. Then we find

$$\begin{aligned}\not{s}_f^2 a_{w+\frac{1}{2}} =& A_0 \frac{d}{d\sigma}\left(f(-f a_{w_0+\frac{1}{2}})' + 2w_0 f'(-f a_{w_0+\frac{1}{2}})\right) \\ &+ f(\not{s}_f a_w)' + 2wf'\not{s}_f a_w\end{aligned} \tag{4.40}$$

It is easy to see by substituting from (4.37) that this cannot be satisfied unless A_0=0.

2) $w - w_0$ is half integral ($w = w_0 + n - \frac{1}{2}$), and the transformation of a_w involves a_{w_0}. The details of this case are similar to case 1. We find

$$\not{s}_f a_w = \sum_{m=1}^{n} A_m \frac{d^m f}{d\sigma^m}\frac{d^{n-m}}{d\sigma^{n-m}} a_{w_0} + f a_{w+\frac{1}{2}} \tag{4.41}$$

and

$$f\not{s}_f a_{w+\frac{1}{2}} = \sum_m A_m \frac{d^m f}{d\sigma^m}\frac{d^{n-m}}{d\sigma^{n-m}}\left(f a_{w_0+\frac{1}{2}}\right) + f^2 a_w' + 2wff' a_w \tag{4.42}$$

requiring $A_m = 0$ except for $m = 0$ and $m = n$; then

$$\begin{aligned}\not{s}_f^2 a_{w+\frac{1}{2}} =& A_0 \frac{d^n}{d\sigma^n}\left(-f^2 a_{w_0}' - 2w_0 ff' a_{w_0}\right) + A_n \frac{d^n f}{d\sigma^n}\left(-f a_{w_0}' - 2w_0 f' a_{w_0}\right) \\ &+ f\frac{d}{d\sigma}\left(A_0 \frac{d^n}{d\sigma^n}(f a_{w_0+\frac{1}{2}}) + A_n \frac{d^n f}{d\sigma^n} a_{w_0+\frac{1}{2}}\right) \\ &+ 2wf'\left(A_0 \frac{d^n}{d\sigma^n}(f a_{w_0+\frac{1}{2}}) + A_n \frac{d^n f}{d\sigma^n} a_{w_0+\frac{1}{2}}\right)\end{aligned} \tag{4.43}$$

which again cannot be satisfied unless A_0 and A_n are zero.

We have shown that the lowest weight fields are parts of doublets which decouple from all other fields under super-reparametrizations. One may apply the same procedure to what remains, again and again until the whole representation is reduced to doublets. So any arbitrary representation in terms of covariant quantities may be reduced to doublets.

We have demonstrated above that all representations with covariant components are doublets with weight $(w,\ w+\frac{1}{2})$. There exist other types of representations with components that transform like gauge fields, *i.e.* non-covariantly.[36] Such representations as well as the covariant derivatives can be constructed, using techniques introduced in the bosonic case. Let $\mathcal{F}$ be a 2 x 2 matrix of operators acting on a doublet (a,b) of weight w_a, and producing a doublet (A,B) of weight w_A:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.44}$$

Application of the doublet transformation laws then yields the following transformation equations for the matrix elements of $\mathcal{F}$:

$$\begin{aligned}
\oint_f F_{11} &= \mp F_{12} f \frac{d}{d\sigma} \mp 2w_a F_{12} \frac{df}{d\sigma} - f F_{21} \\
\oint_f F_{22} &= - f \frac{dF_{12}}{d\sigma} - f F_{12} \frac{d}{d\sigma} - 2w_A \frac{df}{d\sigma} F_{12} \mp F_{21} f \\
\oint_f F_{12} &= \pm F_{11} f - f F_{22} \\
\oint_f F_{21} &= (\pm F_{22} f - f F_{11}) \frac{d}{d\sigma} - f \frac{dF_{11}}{d\sigma} \pm 2w_a F_{22} \frac{df}{d\sigma} - 2w_A \frac{df}{d\sigma} F_{11}
\end{aligned} \tag{4.45}$$

The upper signs in these equations are to be read when F_{12} and F_{21} are anticommuting operators, and the lower signs when F_{11} and F_{22} are the anti-commuting operators. Since application of another such matrix to the doublet (A,B) must yield yet another doublet (C,D), the matrix product of two $\mathcal{F}$ matrices must satisfy the same transformation law (4.45).

As $\mathcal{F}$ is allowed to contain derivatives, the ordering of products in these equations is important even classically. These equations have many solutions, depending on the number of derivatives present in the F's. In the following we discuss several simple cases; the general case is presented in Appendix A.

First, assume that $\mathcal{F}$ contains no derivatives. We immediately deduce that $F_{12} = F_{11} \mp F_{22} = 0$. (F_{11}, F_{21}) transform as a doublet, yielding a way of composing two doublets (a, b) and (c, d) to make another doublet which we recognize as the $\otimes_a$ coupling scheme. We can rewrite this in matrix form as

$$\begin{pmatrix} ac & 0 \\ bc \pm ad & ac \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & \pm a \end{pmatrix} \begin{pmatrix} c & 0 \\ d & \pm c \end{pmatrix}. \tag{4.46}$$

As before, if $w_a + w_c = \frac{1}{2}$, the integral of the heavy component of the compound doublet is an invariant. One can also use Grassmann notation with $\theta = \sigma_-$ identified as the nilpotent component.

We now investigate representations built from $\mathcal{F}$ operators containing at most first derivatives:

$$\mathcal{F} = \mathcal{G} + \mathcal{H}\frac{d}{d\sigma}. \tag{4.47}$$

We consider only the case in which the off-diagonal components of $\mathcal{F}$ are anticommuting. From (4.45) we find

$$H_{12} = 0 \qquad \text{and} \qquad H_{11} = H_{22} \tag{4.48}$$

and

$$\begin{aligned}
\oint_f H_{11} &= -f(H_{21} + G_{12}) \\
\oint_f H_{21} &= f(G_{22} - G_{11} - H'_{11}) + 2(\Delta + \frac{1}{2})f' H_{21} \\
\oint_f G_{11} &= -f G_{21} - 2w_a f' G_{12} \\
\oint_f G_{22} &= -f G'_{12} - f G_{21} - f'(2w_A G_{12} + H_{21}) \\
\oint_f G_{12} &= f(G_{11} - G_{22}) + f' H_{11} \\
\oint_f G_{21} &= -f G'_{11} + 2f'(w_a G_{22} - w_A G_{11}) + 2w_a f'' H_{11}.
\end{aligned} \tag{4.49}$$

Under reparametrizations, some of these fields transform covariantly and some anomalously. It is easy to see that H_{11} has weight $\Delta - 1$, G_{12} and H_{21}

have weight $\Delta - \frac{1}{2}$; all of these fields are covariant. However,

$$\begin{aligned}
\delta_f G_{11} &= D_f^{(\Delta)} G_{11} + w_a f'' H_{11} \\
\delta_f G_{22} &= D_f^{(\Delta)} G_{22} + (w_a + \frac{1}{2}) f'' H_{11} \\
\delta_f G_{21} &= D_f^{(\Delta+\frac{1}{2})} G_{21} + w_a f'' H_{21}
\end{aligned} \tag{4.50}$$

where $D_f^{(w)} \equiv -(f\frac{d}{d\sigma} + w\frac{df}{d\sigma})$. The two fields H_{11} and $H_{21}+G_{12}$ transform into one another as a doublet. The other four fields transform into these fields, so it would seem that the representation is irreducible. However, except for rather special values of the parameters, it is possible to find linear combinations of the fields and their derivatives whose transformation laws decouple into doublets. Specifically,

$$(2\Delta - 1)G_{12} + H_{21} \qquad \text{and} \qquad 2(\Delta - 1)(G_{11} - G_{22}) - \frac{dH_{11}}{d\sigma} \tag{4.51}$$

form a doublet. Clearly, for $\Delta = 1$, this doublet is not independent of the first doublet. The combinations

$$(1-2w_A)G_{11} + 2w_a(G_{22} - H'_{11}) \qquad \text{and} \qquad -2w_a H'_{21} + (1-2\Delta)G_{21} \tag{4.52}$$

form a third new doublet, except for $\Delta = \frac{1}{2}$. Thus, except for these two values of Δ, this six-field representation can be reduced into three doublets. However, for $\Delta = \frac{1}{2}$, if it is also true that $w_a = 0$, then this sextet representation can still be reduced into three doublets, given by

$$(H_{11}, H_{21}+G_{12}), \qquad (G_{11}, G_{21}) \qquad \text{and} \qquad (H_{21}, G_{22}-G_{11}-H'_{11}). \tag{4.53}$$

However, whenever $w_a \neq 0$, we know that the transformation of G_{21} under reparametrizations has a non-covariant term proportional to H_{21}, which has zero weight when $\Delta = \frac{1}{2}$. Thus it is impossible to cancel this anomalous term by adding a derivative of H_{21} to G_{21}. On the other hand, components of

the doublet transform covariantly under reparametrizations, which leads us to conclude that it is not possible to split the sextet into doublets in this case. Rather, the sextet splits into a doublet and a quartet. Its members are given by

$$
\begin{aligned}
(\overline{G}_{21}, \overline{G}_{11}, \overline{G}_{22}, \overline{G}_{12}) \equiv (G_{21} - 2w_a G'_{12}, (2w_a + 1)G_{11} - 2w_a G_{22},& \\
2w_a G_{11} - (2w_a - 1)G_{22} - H'_{11}, -H_{21}),&
\end{aligned}
$$

with the transformation laws

$$
\begin{aligned}
\not\delta_f \overline{G}_{21} &= -f\overline{G}'_{11} - f'[(2w_a + 1)\overline{G}_{11} - 2w_a \overline{G}_{22}] \\
\not\delta_f \overline{G}_{11} &= -f\overline{G}_{21} - 2w_a f'\overline{G}_{12} \\
\not\delta_f \overline{G}_{22} &= -f\overline{G}_{21} - f\overline{G}_{12} - 2w_a f\overline{G}_{12} \\
\not\delta_f \overline{G}_{12} &= f(\overline{G}_{11} - \overline{G}_{22}).
\end{aligned}
\tag{4.54}
$$

Under reparametrizations, $\overline{G}_{11}$ and $\overline{G}_{22}$ have weight $\frac{1}{2}$, $\overline{G}_{12}$ has weight zero, and $\overline{G}_{21}$ has weight one. All of these fields transform covariantly except for $\overline{G}_{21}$, which transforms as

$$
\delta_f \overline{G}_{21} = D_f^{(1)} \overline{G}_{21} + w_a f'' \overline{G}_{12}. \tag{4.55}
$$

This quartet representation of the super-reparametrization algebra is irreducible. Finally, when $\Delta = 1$, with $w_a \neq 0$, we obtain the quartet with slightly different transformation laws, namely

$$
\begin{aligned}
(\hat{G}_{21}, \hat{G}_{11}, \hat{G}_{22}, \hat{G}_{12}) \equiv (G_{21} - 2w_a G'_{12}, G_{11} - 2w_a(G_{11} - G_{22}),& \\
G_{22} - H'_{11}, 2w_a G_{12} - H_{21}),&
\end{aligned}
$$

The 'hatted' fields have the transformations

$$
\begin{aligned}
\not\delta_f \hat{G}_{21} &= -f\hat{G}'_{11} + f'(2w_a \hat{G}_{22} - \hat{G}_{11}) \\
\not\delta_f \hat{G}_{11} &= -f\hat{G}_{21} - 2w_a f' \hat{G}_{12} \\
\not\delta_f \hat{G}_{22} &= -f\hat{G}'_{12} - f' \hat{G}_{12} \\
\not\delta_f \hat{G}_{12} &= f(\hat{G}_{11} - \hat{G}_{22}).
\end{aligned}
\tag{4.56}
$$

These two representations can be understood as special cases of the generic quartet obtained by setting

$$H_{11} = H_{21} + G_{12} = 0$$

in the sextet transformation laws. In general, for representations with more derivatives, it is not possible to completely reduce the representation into doublets, as we shall see in Appendix A.

We conclude this section with the building of the covariant derivative which is the direct generalization of the one we have constructed in the bosonic case. Our starting point will be the quartet with $\Delta = \frac{1}{2}$, and with the off-diagonal elements behaving as fermions, because the derivative operator appears only below the diagonal. As this involves some changes of signs from the above, we repeat the transformation laws of the quartet:

$$\begin{aligned}
\delta_f G_{11} &= -fG_{21} - 2wf'G_{12} \\
\delta_f G_{22} &= -fG_{21} - 2wf'G_{12} - fG'_{12} \\
\delta_f G_{12} &= f(G_{11} - G_{22}) \\
\delta_f G_{21} &= -(fG_{11})' + 2wf'(G_{22} - G_{11}).
\end{aligned} \tag{4.57}$$

All components except G_{21} transform covariantly, with weights $(\frac{1}{2}, \frac{1}{2}, 0, 1)$, respectively. Let us define the new constructs

$$A \equiv \frac{G_{21}}{wG_{12}}; \;\; \chi \equiv \frac{G_{11}}{wG_{12}}; \; \zeta \equiv \frac{(G_{11} - G_{22})}{G_{12}}; \; D \equiv \ln G_{12},$$

in terms of which the transformation laws read (using $\zeta^2 = \chi^2 = 0$)

$$\begin{aligned}
\delta_f D &= f\zeta, \\
\delta_f \zeta &= fD', \\
\delta_f \chi &= -fA - 2f' + f\chi\zeta, \\
\delta_f A &= -(f\chi)' - 2f'\zeta - f\chi D' - fA\zeta.
\end{aligned} \tag{4.58}$$

These are non-linear, but A transforms exactly like the bosonic connection.

It is interesting to note that the point $D = \text{constant}, \zeta = 0$ is stable under super-reparametrizations, leaving us with the anomalous doublet

$$\begin{aligned} \not\delta_f \chi &= -fA - 2f', \\ \not\delta_f A &= -(f\chi)'. \end{aligned} \tag{4.59}$$

Since A transforms as a total derivative, one can then identify A with the derivative of the bosonic ϕ field. It is not possible to build an anomaly free representation of the reparametrization algebra with a suitable integration measure by just using this doublet.[36] We shall see later that it is necessary to use two such doublets for this purpose.

4.3 Construction of a Dynamical Invariant

As mentioned earlier, the string coordinates transform into the generalized gamma matrices under a super-reparametrization. We can separate $\Gamma^\mu(\sigma)$ and $\frac{\delta}{\delta\Gamma^\mu(\sigma)}$ into left and right-moving parts as we did for the coordinates:

$$\Gamma_L^\mu(\sigma) = \Gamma^\mu(\sigma) + i\frac{\delta}{\delta\Gamma_\mu(\sigma)} \tag{4.60}$$

$$\Gamma_R^\mu(\sigma) = \Gamma^\mu(\sigma) - i\frac{\delta}{\delta\Gamma_\mu(\sigma)} \tag{4.61}$$

These of course transform into the left and right-moving parts of the coordinates:

$$\not\delta_f \Gamma_L^\mu = -f x'^\mu_L \tag{4.62}$$

$$\not\delta_f \Gamma_R^\mu = -f x'^\mu_R \tag{4.63}$$

They satisfy the commutations

$$\{\Gamma_L^\mu(\sigma_1), \Gamma_L^\nu(\sigma_2)\} = i g^{\mu\nu}\delta(\sigma_1 - \sigma_2) \tag{4.64}$$

$$\{\Gamma_R^\mu(\sigma_1), \Gamma_R^\nu(\sigma_2)\} = ig^{\mu\nu}\delta(\sigma_1 - \sigma_2) \tag{4.65}$$

$$\{\Gamma_L^\mu(\sigma_1), \Gamma_R^\nu(\sigma_2)\} = 0. \tag{4.66}$$

Here the delta function on the right hand side is defined over the interval $[-\pi, \pi]$. We note that (x_L^μ, Γ_L^μ) and (Γ_L^μ, x'^μ_L) are both doublets. The latter is more useful since it is translationally invariant. The generator of super-reparametrizations for these fields is then

$$\mathcal{M}_f^L = -\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} f \Gamma_L \cdot x'_L \tag{4.67}$$

and similarly

$$\mathcal{M}_f^R = -\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} f \Gamma_R \cdot x'_R. \tag{4.68}$$

We note that with our normalization for the gamma matrices, $\sqrt{-i}\mathcal{M}^L(\sigma)$ is hermitian. The operators $\mathcal{M}_f$ satisfy the classical algebra

$$\{\mathcal{M}_f, \mathcal{M}_g\} = -2iM_{fg} \tag{4.69a}$$

and similarly

$$\{\mathcal{M}_f^L, \mathcal{M}_g^L\} = -2iM_{fg}^L \tag{4.69b}$$

Also, we have

$$[M_f^L, \mathcal{M}_g^L] = i\mathcal{M}_{fg'-f'g/2}^L. \tag{4.69c}$$

Here the operator M_f^L now includes contributions from the Γ's (as given in (4.4)):

$$M_f^L = -\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} f(\sigma) \left(x'^2 + \Gamma'_L \cdot \Gamma_L \right) \tag{4.70}$$

While (4.69a) and (4.69c) are fine at the quantum level, (4.69b) picks up an anomaly upon quantization. For a general representation, it is easy to show that the anomaly in the algebra of M_f^L with M_g^L takes the form

$$[M_f^L, M_g^L] = iM_{fg'-f'g}^L + \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} (Af'''g + Bf'g), \tag{4.71}$$

where A and B are constants which depend on the representation. The anomaly $C_{f,g}$ in the anticommutation relation

$$\{\mathcal{M}_f^L, \mathcal{M}_g^L\} = -2iM_{fg} + C_{f,g} \tag{4.72}$$

can be related to A and B through the Jacobi identity. Specifically, the identity (the superscript L has been suppressed)

$$[\{\mathcal{M}_f, \mathcal{M}_g\}, M_h] + \{[\mathcal{M}_g, M_h], \mathcal{M}_f\} - \{[M_h, \mathcal{M}_f], \mathcal{M}_g\} = 0 \tag{4.73}$$

tells us that $C_{f,g}$ must be

$$C_{f,g} = -\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi}(4Afg'' + Bfg). \tag{4.74}$$

It can be seen from the commutations that classically, $(\mathcal{M}^L(\sigma), 2M(\sigma))$ forms a covariant weight (3/2,2) doublet, but the covariance is spoiled due to quantum ordering effects.

How can we form an invariant dynamical operator which yields consistent equations of motion? We want invariance under reparametrizations as well as super-reparametrizations, so we would like to construct the dynamical invariant as the integral of the heavier component of a weight (1/2,1) doublet.[36] Starting from the above $(\mathcal{M}^L, 2M^L)$ doublet, if we could restore covariance, we could multiply it by a $(-1,-1/2)$ doublet to get a (1/2,1) doublet. We recall that the field $c_L = e^{\phi_L}$ that we had before was a weight -1 field. So we define its partner γ_L to be a weight one-half field:

$$\not{\delta}_f c_L = -f\gamma_L = -i\{\mathcal{M}_f, c_L\} \tag{4.75}$$

$$\not{\delta}_f \gamma_L = -(fc' - 2f'c) \tag{4.76}$$

Henceforth we shall, for convenience, drop the sub(super)script 'L'; it will be understood that all fields(unless otherwise mentioned) have this sub(super)script.

Here the generator for this super-reparametrization can be written in the form

$$\mathcal{M}_f^{gh} = i\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi}\left(-f\gamma\frac{\delta}{\delta c} - (fc' - 2f'c)\frac{\delta}{\delta\gamma}\right) \tag{4.77}$$

Now $b = e^{-\phi}$ is conjugate to c, so $\frac{\delta}{\delta c}$ is simply the field b. Similarly, the field $\frac{\delta}{\delta\gamma}$ is the field conjugate to γ, which we shall call β. Since γ has weight -1/2, β must have weight 3/2. Also, b has weight two. Therefore (β, b) is a (3/2,2) doublet pair:

$$\not{\delta}_f \beta = -fb \tag{4.78}$$

$$\not{\delta}_f b = -(f\beta' + 3f'\beta) \tag{4.79}$$

and

$$[\beta(\sigma), \gamma(\sigma')] = \delta(\sigma - \sigma') \tag{4.80}$$

The fields β and γ have the mode expansions

$$\gamma(\sigma) = \sqrt{-i}\sum \gamma_n e^{in\sigma} \tag{4.81}$$

$$\beta(\sigma) = \sqrt{i}\sum \beta_n e^{in\sigma} \tag{4.82}$$

The modes γ_n and β_n are hermitian and anti-hermitian respectively and satisfy

$$[\beta_n, \gamma_m] = \delta_{m+n} \tag{4.83}$$

Then $\mathcal{M}_f^{gh}$ can be written in the form

$$\mathcal{M}_f^{gh} = -i\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} f(\gamma b + 3c'\beta + 2c\beta') \tag{4.84}$$

The corresponding reparametrization generator for the ghosts can of course be obtained by anticommuting two $\mathcal{M}$ operators; apart from a c-number anomaly,

$$\{\mathcal{M}_f^{gh}, \mathcal{M}_g^{gh}\} = -2iM_{fg}^{gh} \tag{4.85}$$

where

$$M_f^{gh} = i\int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}f(cb' + 2c'b + \frac{1}{2}\gamma\beta' + \frac{3}{2}\beta\gamma') \tag{4.86}$$

The total anomaly in the algebra of $\mathcal{M}_f^{tot} = (\mathcal{M}_f^{x,\Gamma} + \mathcal{M}_f^{gh})$ is proportional to

$$c = \frac{3}{2}d - 2(6w_b^2 - 6w_b + 1) + 2(6w_\beta^2 - 6w_\beta + 1) \tag{4.87}$$

and so cancels in ten spacetime dimensions.

Now we would like to construct an invariant dynamical Lorentz scalar operator for use in our Lagrangian. We want to construct it, as mentioned earlier, as the integral of the heavy component of a (1/2,1) doublet. It turns out, however, that the heavier component of the product

$$(c, \gamma) \otimes_a (\mathcal{M}_{tot}, 2M_{tot}) = (c\mathcal{M}_{tot}, 2cM_{tot} - \gamma\mathcal{M}_{tot}) \tag{4.88}$$

is not covariant upon normal-ordering, despite the fact that the total M and $\mathcal{M}$ operators are now anomaly-free. This arises because of additional ordering ambiguities in the product (4.88). It turns out that the correct prescription is to include only half the naive ghost contribution to $(\mathcal{M}, 2M)$. Then the invariant dynamical operator we have is[39,36]

$$\mathcal{Q} = \int_{-\pi}^{\pi}\frac{d\sigma}{2\pi}\left[(\gamma\mathcal{M}_{x,\Gamma}(\sigma) - 2cM_{x,\Gamma}(\sigma)) + \frac{1}{2}(\gamma\mathcal{M}_{gh}(\sigma) - 2cM_{gh}(\sigma))\right] \tag{4.89}$$

This hermitian operator is invariant upon overall normal-ordering, and is simply the nilpotent BRST charge of superstring theory. Again, nilpotency here turns out to be a property of the invariant; we do not require it at the outset, but end up with it anyway. The free action constructed from this invariant has the simple form

$$S = <\Psi|\mathcal{Q}|\Psi> . \tag{4.90}$$

We note that this action is second-order in time derivatives, unlike the usual action for a fermion. The supersymmetry of the theory mixes first and second-order operators, and it is therefore necessary to include them both in the construction of the dynamical invariant. However, it can be shown that the gauge-fixed form of this action is indeed first-order. In the next chapter we shall compile a list of invariants and 'fermionize' the superghosts.

CHAPTER 5
VECTOR AND TENSOR INVARIANTS

5.1 Invariants in the standard representation

One may ask what other invariants it is possible to construct in the bosonic and supersymmetric theories. In the case of the bosonic string, the following objects[36] are invariants:

(1) The momentum vector

$$p_\mu \equiv -i \int_0^\pi \int_0^\pi \frac{d\sigma}{\pi} \frac{\delta}{\delta x^\mu(\sigma)}.$$

(2) The ghost number

$$N_G \equiv -i \int_0^\pi \int_0^\pi \frac{d\sigma}{\pi} \frac{\delta}{\delta \phi(\sigma)}.$$

(3) The Lorentz generators

$$M_{\mu\nu} \equiv i \int_0^\pi \int_0^\pi \frac{d\sigma}{\pi} \left(x_\mu \frac{\delta}{\delta x^\nu} - x_\nu \frac{\delta}{\delta x^\mu} \right)$$

.

(4) The symmetric space-time tensor

$$Q^{\mu\nu} \equiv \int_{-\pi}^\pi \int_0^\pi \frac{d\sigma}{\pi} : e^{\phi_L} \left[x_L'^\mu x_L'^\nu - \frac{g^{\mu\nu}}{26} (\phi_L'^2 + 3\phi_L'') \right] : \ .$$

The BRST operator is obtained by taking the trace of the above symmetric tensor:

$$Q = g_{\mu\nu} Q^{\mu\nu}. \tag{5.1}$$

This invariant tensor depends on the space-time geometry. Its most interesting property is that its spacetime trace is the BRST operator. We shall shortly look at the algebra satisfied by this tensor. We note that like the BRST charge, this tensor is a ghost number one object. The algebra of this tensor generates another symmetric tensor, as we shall see. The results of this chapter are based on the work of ref.[23,36].

We note that although the dilatation operator $D = \int \int_0^\pi \frac{d\sigma}{\pi} : x \cdot \frac{\delta}{\delta x} :$ has the right weight to be a classical invariant, it transforms anomalously due to ordering effects. Thus the largest space-time symmetry seems to be that of the Poincaré group. We remark that there does not exist an invariant 26-vector which serves as the string position in space-time. This is not too surprising since the theory is not (space-time) conformally invariant. On the other hand, by specializing the Poincaré generators to the relevant space-like surfaces, we can define a physical position for the string in 25 (at equal time) or 24 (light cone) space dimensions.

One can now look for a bigger list of invariants in the supersymmetric theory. It is possible to construct invariants in the supersymmetric theory by combining the various doublets present with one another according to (4.30). The fundamental doublets present in the theory are

$$\begin{aligned} (x, \Gamma)^\mu &\quad \text{with weight } (0, \tfrac{1}{2}), \\ (\Gamma, x')^\mu &\quad \text{with weight } (\tfrac{1}{2}, 1), \\ (e^\phi, \gamma) &\quad \text{with weight } (-1, -\tfrac{1}{2}), \text{ and} \\ (\beta, e^{-\phi}) &\quad \text{with weight } (\tfrac{3}{2}, 2). \end{aligned} \tag{5.2}$$

As before, we leave it understood that all fields represent left movers only, and that all exponentials of fields are implicitly normal ordered. In taking

products of such exponentials, the normal ordering must be carefully taken into account. Using the Baker-Hausdorf identity

$$e^A e^B = e^{A+B} e^{[A,B]/2}$$

which is true for any operators A and B which commute with their commutator, we find

$$: e^{a\xi(\sigma_1)} :: e^{b\xi(\sigma_2)} :=: e^{a\xi(\sigma_1+i\epsilon)+b\xi(\sigma_2)} : \left[-2i\sin\left(\frac{\sigma_1-\sigma_2+i\epsilon}{2}\right)\right]^{-\eta ab} \tag{5.3}$$

where ϵ is a small positive number needed for convergence, and η is the sign of the commutator of the modes of the field ξ (see eqn. (3.76)). From this we see that if $\eta = -1$,

$$\{e^{\xi(\sigma_1)}, e^{-\xi(\sigma_2)}\} = \delta(\sigma_1 - \sigma_2) \tag{5.4}$$

First consider looking for invariants made up of the product of two of these doublets. We have seen that such an invariant must be constructed from the form (4.30a). To use $(\Gamma, x')^\mu$ we would need to combine it with a doublet of weight $(0, \frac{1}{2})$; the only such thing here is $(x, \Gamma)^\mu$, and this combination produces a trivial invariant. The other two doublets (e^ϕ, γ) and $(\beta, e^{-\phi})$ have the right weight to be combined and yield an invariant. The invariant so constructed has the form

$$\int\int_0^\pi \frac{d\sigma}{\pi}(e^\phi, \gamma) \otimes_a (\beta, e^{-\phi}) = \int_{-\pi}^\pi \int_0^\pi \frac{d\sigma}{\pi}\left(-e^\phi e^{-\phi} + \gamma\beta\right) \tag{5.5}$$

which we recognize as the ghost number (the right-hand-side above is understood to be normal ordered).

Next we may look for further invariants by taking products of three doublets.[36] These may be constructed by taking any two of the above doublets together according to any of the three product rules (4.30), then combining the result

of this with another doublet according to (4.30a) (the other two would yield trivial invariants) in such a way as to achieve a final result with weight $(\frac{1}{2}, 1)$. Note that such triple products do not in general satisfy associativity. With the four doublets present, there are 192 possible combinations, 12 of them with the proper weight to be invariants. These fall into three categories. These we now list:

(1) Products which give zero upon integration. These are:

$$\{(x, \Gamma) \otimes_b (x, \Gamma)\} \otimes_a (x, \Gamma)$$

$$\{(x, \Gamma) \otimes_a (\Gamma, x')\} \otimes_a (x, \Gamma)$$

and

$$\{(x, \Gamma) \otimes_a (x, \Gamma)\} \otimes_a (\Gamma, x')$$

(2) Certain products involving only the ghost fields. These are:

$$\{(e^{\phi}, \gamma) \otimes_c (\beta, e^{-\phi})\} \otimes_a (e^{\phi}, \gamma)$$

$$\{(e^{\phi}, \gamma) \otimes_c (e^{\phi}, \gamma)\} \otimes_a (\beta, e^{-\phi})$$

These two products are identical when evaluated. They reproduce the part of the BRST charge (4.89) involving ghosts only, which we will denote by $\mathcal{Q}_{ghost}$

$$\begin{aligned} \mathcal{Q}_{ghost} = i \int \int_0^{\pi} \frac{d\sigma}{\pi} : [\gamma^2 e^{-\phi} + 2\gamma\beta' e^{\phi} + 3\gamma\beta(e^{\phi})' \\ - e^{\phi}(\phi'^2 + 3\phi'' + \gamma\beta' + 3\beta\gamma')] : \end{aligned} \tag{5.6}$$

However, this quantity by itself is not invariant after overall normal ordering.

(3) Certain products involving both ghost and coordinate fields, which yield a second rank tensor. These are:

$$\{(\Gamma, x')^{\mu} \otimes_c (e^{\phi}, \gamma)\} \otimes_a (x, \Gamma)^{\nu}$$

$$\{(x,\Gamma)^\mu \otimes_c (e^\phi,\gamma)\} \otimes_a (\Gamma, x')^\nu$$

$$\{(\Gamma,x')^\mu \otimes_b (e^\phi,\gamma)\} \otimes_a (\Gamma, x')^\nu$$

$$\{(x,\Gamma)^\mu \otimes_c (\Gamma,x')^\nu\} \otimes_a (e^\phi,\gamma)$$

These four expressions are identical up to total derivatives, and so lead to the same invariant; the result is a second-rank tensor

$$\tilde{Q}^{\mu\nu} = \int_{-\pi}^{\pi}\int_0^{\pi} \frac{d\sigma}{\pi} : \left[e^\phi(x'^\mu x'^\nu + \Gamma'^\mu\Gamma^\nu) - \gamma\Gamma^\nu x'^\mu + (\mu \longleftrightarrow \nu)\right] : . \tag{5.7}$$

It is the supersymmetric generalization of the bosonic invariant metric-like tensor we have previously discussed. The diagonal elements of this tensor transform anomalously; however, in $d = 10$ it is possible to form the anomaly-free combination

$$Q^{\mu\nu} = \tilde{Q}^{\mu\nu} + \frac{1}{10} g^{\mu\nu} Q_{ghost} \tag{5.8}$$

As in the bosonic case, taking the trace of this tensor operator reproduces the BRST charge:[36]

$$Q = Q^{\mu\nu} g_{\mu\nu} \tag{5.9}$$

5.2 Algebra of the bosonic string tensor invariants

The bosonic string tensor invariants $Q^{\mu\nu}$ satisfy an interesting algebra. We recall that this tensor was given by the expression

$$Q^{\mu\nu} \equiv \int_{-\pi}^{\pi}\int_0^{\pi} \frac{d\sigma}{\pi} : e^{\phi_L}\left[x_L'^\mu x_L'^\nu - \frac{g^{\mu\nu}}{26}(\phi_L'^2 + 3\phi_L'')\right] : .$$

This symmetric spacetime tensor is an anticommuting operator of ghost number one. Its algebra with the BRST charge Q is[23]

$$\{Q, Q^{\mu\nu}\} = -2iB^{\mu\nu} \tag{5.10}$$

where the symmetric tensor $B^{\mu\nu}$ is defined by

$$\begin{aligned} B^{\mu\nu} &= \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} : cc'(x'^{\mu}x'^{\nu} - \frac{g^{\mu\nu}}{26}x' \cdot x') : \\ &= i\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} : e^{2\phi}(x'^{\mu}x'^{\nu} - \frac{g^{\mu\nu}}{26}x' \cdot x') : \end{aligned} \tag{5.11}$$

The tensor $B^{\mu\nu}$ has ghost number two. It is easy to check that $B^{\mu\nu}$ is indeed invariant under reparametrizations. Further, since the BRST charge is the trace of $Q^{\mu\nu}$, the nilpotency of Q ensures that $B^{\mu\nu}$ is traceless; the statement that $B^{\mu\nu}$ is traceless is equivalent to the statement that Q is nilpotent. We note that $B^{\mu\nu}$ commutes with $Q^{\rho\sigma}$ and therefore also with the BRST charge. Also,

$$[B^{\mu\nu}, B^{\rho\sigma}] = 0. \tag{5.12}$$

The algebra of the components of $Q^{\mu\nu}$ with themselves is more complicated. One finds

$$\begin{aligned} \{Q^{\mu\nu}, Q^{\rho\sigma}\} = &-i(g^{\nu\rho}B^{\mu\sigma} + g^{\mu\rho}B^{\nu\sigma} + g^{\mu\sigma}B^{\nu\rho} + g^{\nu\sigma}B^{\mu\rho}) \\ &+ \frac{i}{13}[(g^{\mu\nu}B^{\rho\sigma} + g^{\rho\sigma}B^{\mu\nu}) - (g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma} - \frac{1}{13}g^{\mu\nu}g^{\rho\sigma})C] \end{aligned} \tag{5.13}$$

where C is the object

$$C = \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} cc'x'_L \cdot x'_L = i\int_{-\pi}^{\pi} \frac{d\sigma}{2\pi} e^{2\phi_L}x'_L \cdot x'_L. \tag{5.14}$$

This object is invariant, but it is not normal-ordered. In fact, it has operator anomalies upon normal-ordering. As a means of projecting out the anomalous

part of the algebra, we can introduce a set of 26 x 26 matrices $\alpha^I_{\mu\nu}$ such that the projections[23]

$$Q^I \equiv \alpha^I_{\mu\nu} Q^{\mu\nu} = Tr(\alpha^I \mathbf{Q}) \tag{5.15}$$

obey a non-anomalous algebra, where the trace is taken over spacetime indices. Then it can be easily checked that the Q^I's obey the non-anomalous algebra

$$\{Q^I, Q^J\} = -2i\{\alpha^I, \alpha^J\}_{\mu\nu} B^{\mu\nu} + \frac{i}{13}(B^I \mathrm{Tr}(\alpha^J) + B^J \mathrm{Tr}(\alpha^I)) \tag{5.16}$$

provided that the α's satisfy

$$\mathrm{Tr}(\alpha^I \alpha^J) = \frac{1}{26} \mathrm{Tr}(\alpha^I) \mathrm{Tr}(\alpha^J). \tag{5.17}$$

We note that the spacetime metric itself satisfies this equation. By taking out the trace part of these matrices, this condition becomes equivalent to the requirement

$$\mathrm{Tr}\alpha^I = \mathrm{Tr}(\alpha^I \alpha^J) = 0. \tag{5.18}$$

The number p of such independent matrices in d dimensions can be determined from the obvious relation

$$p = \frac{d(d+1)}{2} - 1 - \frac{p(p+1)}{2} \tag{5.19}$$

which yields $p = d - 1$. So there are exactly 25 matrices in every such set in 26 dimensions. Of course, there is an infinite number of such sets that one can construct. It is easier to work with α matrices with one covariant and one contravariant index, since then the trace is the usual sum of diagonal elements and the matrix multiplication is easy to do. Then the α's have the standard form

$$\alpha^\mu{}_\nu = \begin{pmatrix} -\mathrm{Tr}\mathbf{A} & \mathbf{a}^T \\ -\mathbf{a} & \mathbf{A} \end{pmatrix} \tag{5.20}$$

where $\mathbf{a}$ is a 25-vector and $\mathbf{A}$ is a symmetric 2525 matrix such that they satisfy the constraints

$$\mathrm{Tr}(\mathbf{A}^I\mathbf{A}^J) + \mathrm{Tr}\mathbf{A}^I\mathrm{Tr}\mathbf{A}^J = 2\mathbf{a}^I \cdot \mathbf{a}^J \tag{5.21}$$

Nilpotency of any one of the α's is equivalent to demanding that

$$\mathbf{A}^2 = \mathbf{a}\mathbf{a}^T \tag{5.22a}$$

and

$$\mathbf{A}\mathbf{a} = (\mathrm{Tr}\mathbf{A})\mathbf{a}. \tag{5.22b}$$

A solution of these equations is in terms of the null vector a_μ which has the components $(\mathrm{Tr}\mathbf{A}, \mathbf{a})$. Then we have the simple relation

$$\alpha_{\mu\nu} = a_\mu a_\nu \tag{5.23}$$

for the components of the nilpotent matrix. It is easy to see that there can exist at most one nilpotent in the set of the α^I's. Such a nilpotent would of course correspond to a nilpotent $Q^{\mu\nu}$-projection Q^I. It would be interesting to look at the cohomology of this nilpotent.

5.3 Fermionization of the Superconformal Ghosts

It is well-known[34] that the superconformal ghosts β and γ can be rewritten in terms of quantities χ, η, and ξ, as follows:

$$\beta = i\xi' e^{-\chi} \tag{5.24}$$

$$\gamma = \eta e^{\chi} \tag{5.25}$$

The commutation relation between the conjugate fields β and γ can then be reproduced if we choose the fermionic fields ξ and η to be conjugate, and if χ is a field whose modes χ_n satisfy the commutations

$$[\chi_n, \chi_m] = +n\delta_{n,-m}. \tag{5.26}$$

This field transforms anomalously with an inhomogeneous term:

$$\delta\chi = -(f\chi' - f'). \tag{5.27}$$

In order for β and γ to transform covariantly with the right weights, η and ξ must have weights 1 and 0 respectively, and e^{χ} and $e^{-\chi}$ must have weights $-\frac{3}{2}$ and $\frac{1}{2}$ respectively. This is true, since the normal-ordered exponential $e^{a\chi}$ transforms covariantly with weight $-a(a+2)/2$. Of course, the anomaly in the supersymmetry algebra still cancels for $d = 10$. Upon investigating the supersymmetry transformations of these fields, we find that the fields themselves (ϕ, χ, ξ and η) form a non-linear representation of the super-reparametrization algebra. However, various combinations of these fields belong to doublet representations.[36] These are as follows:

$$\begin{aligned}
&[e^{\phi}, \eta e^{\chi}] && \text{with weight } (-1, -\frac{1}{2}) \\
&[i\xi' e^{-\chi}, e^{-\phi}] && \text{with weight } (\frac{3}{2}, 2) \\
&[\xi, e^{-\phi} e^{\chi}] && \text{with weight } (0, \frac{1}{2}) \\
&[-i(e^{\phi}(e^{-\chi})' + \frac{1}{2}(e^{\phi})' e^{-\chi}, \eta - (i\xi' e^{-2\chi} e^{2\phi})'] && \text{with weight } (\frac{1}{2}, 1) \\
&[e^{\chi}, \eta e^{-\phi} e^{2\chi}] && \text{with weight } (-\frac{3}{2}, -1).
\end{aligned} \tag{5.28}$$

Given these doublets (the first two pairs are conjugate to one another), one can of course compose them to form further doublets. We note the curious fact

that one can obtain vector invariants from the following two ways of composing doublets:

$$(\xi, e^{-\phi}e^{\chi}) \otimes_a (\Gamma, x')^\mu = [\xi\Gamma, x'\xi - e^{-\phi}e^{\chi}\Gamma]^\mu \tag{5.29}$$

and

$$\begin{aligned}&[-i(e^{\phi}(e^{-\chi})' + \frac{1}{2}(e^{\phi})'e^{\chi}), \eta - (i\xi' e^{-2\chi}e^{2\phi})'] \otimes_a (x, \Gamma)^\mu = \\ &[-ix^\mu(e^{\phi}(e^{-\chi})' + \frac{1}{2}(e^{\phi})'e^{\chi}), x^\mu(\eta - (i\xi' e^{-2\chi}e^{2\phi})') - i\Gamma^\mu(e^{\phi}(e^{-\chi})' + \frac{1}{2}(e^{\phi})'e^{-\chi})]\end{aligned} \tag{5.30}$$

In both cases, the heavier components are vectors with weight one and therefore yield invariants when integrated over σ; these are

$$X^\mu = \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi}(x'^\mu\xi - e^{-\phi}e^{\chi}\Gamma^\mu) \tag{5.31}$$

and

$$Y^\mu = \int_{-\pi}^{\pi} \frac{d\sigma}{2\pi}\left(x^\mu(\eta - (i\xi' e^{-2\chi}e^{2\phi})') - i\Gamma^\mu(e^{\phi}(e^{-\chi})' + \frac{1}{2}(e^{\phi})'e^{-\chi})\right). \tag{5.32}$$

We note that Y^μ transforms like a coordinate under translations; thus an invariant coordinate can be defined in the supersymmetric theory, unlike in the bosonic theory. One can presumably form more of these vector invariants by taking products of more doublets; however, except for the above two, all of these seem to be either total derivatives (and hence trivial), or have anomalies upon overall normal-ordering.

In the next chapter, we shall arrive at a supersymmetric bosonization scheme that uses only fields that form a linear representation of the super-reparametrization algebra.

CHAPTER 6
SUPERBOSONIZATION

6.1 Construction of the ghosts

We have seen that the doublet is the only irreducible representation of the super-reparametrization algebra whose components transform covariantly. However, a field $\phi(\sigma)$ transforming according to (3.45) may be used, along with a weight–1/2 anticommuting field which we will call $s(\sigma)$, to provide another representation:

$$\begin{aligned} \not{\delta}_f \phi &= -fs \\ \not{\delta}_f s &= -(f\phi' + 2wf') \end{aligned} \tag{6.1}$$

Note that this inhomogeneous representation coincides with the usual doublet in the case when $w = 0$; we shall refer to it as the anomalous doublet. As mentioned earlier, it can be easily checked that a single anomalous doublet does not allow for the construction of an invariant dynamical operator, even though it provides for an anomaly-free representation of the super-reparametrization algebra. This is because a single such doublet simply does not have a sufficient number of degrees of freedom; we have seen that the standard representation as well as the previously introduced fermionization required two bosonic and two fermionic degrees of freedom. It is therefore natural to consider the possibility that two such anomalous doublets[46,47,48] might provide a satisfactory representation–clearly they would naively possess the correct number of degrees of freedom. The results of this chapter are based on the author's work in ref.48.

For the above multiplet (ϕ, s), the generators have the form

$$M_f^{\phi,s} = -i\int \frac{d\sigma}{2\pi}\left[(f\phi' + wf')\frac{\delta}{\delta\phi} + (fs' + \frac{f's}{2})\frac{\delta}{\delta s}\right] \tag{6.2a}$$

$$\mathcal{M}_f^{\phi,s} = -i\int \frac{d\sigma}{2\pi}\left[fs\frac{\delta}{\delta\phi} + (f\phi' + 2wf')\frac{\delta}{\delta s}\right] \tag{6.2b}$$

As before, this type of multiplet may be separated into left and right moving pieces, which are defined by the relations

$$\phi = \frac{\phi_R + \phi_L}{2} \tag{6.3a}$$

$$\frac{\delta}{\delta\phi} = i\frac{\eta}{2}(\phi_R' - \phi_L') \tag{6.3b}$$

$$s = \frac{s_R + s_L}{2} \tag{6.3c}$$

$$\frac{\delta}{\delta s} = i\frac{\eta}{2}(s_R - s_L) \tag{6.3d}$$

where $\eta = \pm 1$. All left movers commute with right movers over the interval $[0, \pi]$. The generators M_f and $\mathcal{M}_f$ spilt into pieces containing only one type of mover:

$$M_f = M_f^R + M_f^L \tag{6.4a}$$

$$\mathcal{M}_f = \mathcal{M}_f^R + \mathcal{M}_f^L \tag{6.4b}$$

The generators for the standard doublet can be similarly split into left and right movers only if its weights are $(0, 1/2)$. It is a remarkable fact that the string coordinates x^μ and their superpartners form a multiplet with precisely these weights. For the rest of this chapter we shall only deal with left-moving fields and it is understood that similar remarks hold for right movers.

We saw earlier that the ghosts of bosonic string theory may be described either in terms of the anticommuting variables $b(\sigma)$ and $c(\sigma)$, or the commuting variable $\phi(\sigma)$. The relations between these quantities were as follows:

$$b =: e^{-\phi} : \tag{6.5a}$$

$$c =: e^{\phi} : \tag{6.5b}$$

The exponentials of ϕ satisfy the product relation(since $\eta = -1$ for the field ϕ)

$$: e^{a\phi(\sigma_1)} :: e^{b\phi(\sigma_2)} : = \left(-2i \sin\left(\frac{\sigma_1 - \sigma_2}{2}\right)\right)^{ab} : e^{a\phi(\sigma_1) + b\phi(\sigma_2)} : . \tag{6.6}$$

Using (6.6) we may invert (6.5) as

$$\phi'(\sigma) = -i : c(\sigma)b(\sigma) : \tag{6.7}$$

It is not obvious that the Fock space created by the modes ϕ_n of $\phi(\sigma)$ is isomorphic to that created by the modes of the fermionic ghosts $b(\sigma)$ and $c(\sigma)$. There is a well-known proof[38,49] of this equivalence, using Jacobi's triple product identity to relate the partition functions. Here we give another argument.

In either the fermionic or bosonized ghost representation, the full Fock space may be generated by acting with the Virasoro operators on a certain subspace which is referred to as the *highest weight states*. These are defined to be those states which are annihilated by L_n for all $n > 0$. Acting with the other L's (those with $n < 0$) reproduces the full Fock space. The space of highest weight states is labelled by the eigenvalues of normal-ordered operators which commute with all the L's, i.e. which are reparametrization invariant. Using only the fermionic ghosts, the only such operator is the ghost number, defined as

$$N_G = -\int \frac{d\sigma}{2\pi} : b(\sigma)c(\sigma) : \tag{6.8a}$$

With the bosonized ghosts, the only such operator is the zero mode

$$p_{\phi} = -\int \frac{d\sigma}{2\pi} \phi'(\sigma) \tag{6.8b}$$

These two quantities have the same eigenvalue spectrum, and in fact (6.7) shows that they are actually identical except for a factor of i. Thus the space of highest weight states is the same in both representations.

As we have seen, the ghosts in the supersymmetric theory could be bosonized according to

$$\beta = i\xi' e^{-\chi} \tag{6.9a}$$

$$\gamma = \eta e^{\chi} \tag{6.9b}$$

This bosonization of superghosts does not have supersymmetry in the new variables ϕ, χ, ξ, and η, as mentioned earlier; their transformation laws under supersymmetry are nonlinear. For instance, the field χ' transforms as follows:

$$\delta_f \chi' = i\left(f(i\xi'\phi' e^{\phi-\chi} + \eta e^{\chi-\phi}) + 2if'\xi' e^{\phi-\chi}\right) \tag{6.10}$$

Thus the super-reparametrization invariance of the theory is no longer as simply implemented, and this can be inconvenient for some applications.

An alternate bosonization of the conformal and superconformal ghosts has been introduced[46,47,50] which does not sacrifice the superfield structure of the ghosts. This bosonization is as follows:

$$b = \frac{1}{2} u e^{-a\varphi} \tag{6.11a}$$

$$c = \tilde{u} e^{a\varphi} \tag{6.11b}$$

$$\gamma =: (\tilde{\varphi}' - a\tilde{u}u) e^{a\varphi} : \tag{6.11c}$$

$$\beta = -\frac{1}{2a} e^{-a\varphi} \tag{6.11d}$$

In (6.11), (φ, u) and $(\tilde{\varphi}, \tilde{u})$ are supermultiplets transforming according to (6.1). Under reparametrizations, φ and $\tilde{\varphi}$ transform like bosonized fields, i.e.

with an inhomogeneous term as in (3.45). The two multiplets are defined to be conjugate to each other in the sense that

$$[\varphi'(\sigma_1), \tilde{\varphi}'(\sigma_2)] = 2i\frac{d}{d\sigma_1}\delta(\sigma_1 - \sigma_2) \tag{6.12a}$$

$$\{u(\sigma_1), \tilde{u}(\sigma_2)\} = -2i\delta(\sigma_1 - \sigma_2) \tag{6.12b}$$

$$[\varphi(\sigma_1), \varphi(\sigma_2)] = 0 \tag{6.12c}$$

$$[\tilde{\varphi}(\sigma_1), \tilde{\varphi}(\sigma_2)] = 0 \tag{6.12d}$$

$$\{u(\sigma_1), u(\sigma_2)\} = 0 \tag{6.12e}$$

$$\{\tilde{u}(\sigma_1), \tilde{u}(\sigma_2)\} = 0 \tag{6.12f}$$

At this stage, we have changed our conventions a little; the commutations of the conjugate fields b, c and β, γ now have factors of $-i$ and i respectively due to the commutations (6.12). Because all modes of φ commute among themselves, the exponentials in (6.11) have their classical weight, namely $\pm w_\varphi a$, where w_φ is the coefficient of the inhomogenous term in (3.45). Since the weights of u and $\tilde{u}$ must be $\frac{1}{2}$ because of (6.1), we must have $w_\varphi a = -\frac{3}{2}$. Also, we must have $w_{\tilde{\varphi}} = -a$ in order to maintain covariance of γ. Then all of the ghosts transform with the appropriate weights.

It is easy to partially invert (6.11) to obtain

$$\tilde{u} = -2a\beta c \tag{6.13a}$$

$$\tilde{\varphi}' = -2a : (\gamma\beta + bc) : \tag{6.13b}$$

We recall from the previous chapter that this latter expression is just proportional to the superghost number.

We will now derive (6.11) using the representation theory of the super-reparametrization algebra. Consider a pair of self-conjugate doublets (ω, s) and (ψ, t), with the transformation laws[48]

$$\not{\delta}_f \omega = -fs \tag{6.14a}$$

$$\not{\delta}_f s = -(f\omega' + 2wf') \tag{6.14b}$$

$$\not{\delta}_f \psi = -ft \tag{6.14c}$$

$$\not{\delta}_f t = -(f\psi' + 2vf') \tag{6.14d}$$

where w and v are c-numbers. As before, we consider left-movers only. The fields ω, ψ, s and t have the respective Fourier expansions

$$\omega(\sigma) = \omega_0 - \sigma p_\omega + i \sum_{n\neq 0} \frac{\omega_n}{n} e^{in\sigma} \tag{6.15a}$$

$$\psi(\sigma) = \psi_0 - \sigma p_\psi + i \sum_{n\neq 0} \frac{\psi_n}{n} e^{in\sigma} \tag{6.15b}$$

$$s(\sigma) = \sum s_n e^{in\sigma} \tag{6.15c}$$

$$t(\sigma) = \sum t_n e^{in\sigma} \tag{6.15d}$$

where the modes satisfy

$$[\omega_m, \omega_n] = -m\delta_{m,-n} \tag{6.16a}$$

$$[p_\omega, \omega_m] = i\delta_{m,0} \tag{6.16b}$$

$$[\psi_m, \psi_n] = m\delta_{m,-n} \tag{6.16c}$$

$$[p_\psi, \psi_m] = -i\delta_{m,0} \tag{6.16d}$$

$$\{s_n, s_m\} = -i\delta_{m+n} \tag{6.16e}$$

$$\{t_n, t_m\} = i\delta_{m+n} \tag{6.16f}$$

The choice of signs in the commutations above is necessary in order to reproduce the ghost algebra. The generators for the reparametrizations and the super-reparametrizations can of course be easily written down. With the choice of signs we have made, the total anomaly in the algebra of $\mathcal{M}_f$ with $\mathcal{M}_g$ is proportional to

$$B = \frac{d+2}{8} - w^2 + v^2 \tag{6.17}$$

We now investigate the question of what quantities may be formed with these fields which will transform as the ghosts. The basic covariant doublets are

$$(e^{a\omega}, ase^{a\omega})$$

and

$$(e^{b\psi}, bte^{b\psi}).$$

for any constants a and b. As usual, these doublets may be combined with the rules (4.30) to yield additional covariant doublets. There is no ordering problem at this stage. The results are

$$\begin{pmatrix} e^{a\omega}e^{b\psi} \\ (as+bt)e^{a\omega}e^{b\psi} \end{pmatrix} \tag{6.18a}$$

$$\begin{pmatrix} ab\left[(w+a/2)t-(v-b/2)s\right]e^{a\omega}e^{b\psi} \\ a(w+a/2)e^{a\omega}(e^{b\psi})'-b(v-b/2)(e^{a\omega})'e^{b\psi} \\ +\left[a(w+a/2)+b(v-b/2)\right]abste^{a\omega}e^{b\psi} \end{pmatrix} \tag{6.18b}$$

$$\begin{pmatrix} 2b(v-b/2)(e^{a\omega})'e^{b\psi}-2a(w+a/2)e^{a\omega}(e^{b\psi})'+abste^{a\omega}e^{b\psi} \\ 2ab(v-b/2)se^{a\omega}(e^{b\psi})'-a(2a(w+a/2)+1)(se^{a\omega})'e^{b\psi} \\ +(2b(v-b/2)+1)bt(e^{a\omega})'e^{b\psi}-2ab(w+a/2)e^{a\omega}(te^{b\psi})' \end{pmatrix} \tag{6.18c}$$

The lighter components of these doublets transform with weights $a(w+a/2)+b(v-b/2)$ plus 0, 1/2, and 1, respectively. Since c is anticommuting,

and the lighter component of its multiplet, its multiplet must be (6.18b) for some a, b. This has the right weight if

$$a(w + a/2) + b(v - b/2) = -3/2 \tag{6.19}$$

Then we must have

$$\{c(\sigma_1), c(\sigma_2)\} = 0,$$

which will be true only if

$$a^2 = b^2 \tag{6.20a}$$

and

$$(w + a/2)^2 = (v - b/2)^2. \tag{6.20b}$$

Thus we see $a(w + a/2) = b(v - b/2) = -3/4$. Then (6.17) is satisfied if the number of spacetime dimensions d is ten. We will choose $a = b$. Then (removing overall multiplicative constants)

$$c = (t - s)e^{a\omega}e^{a\psi} \tag{6.21a}$$

$$\gamma = (\psi' - \omega' + 2ast)e^{a\omega}e^{a\psi} \tag{6.21b}$$

We obtain the conjugate doublet (β, b) from (4.30a) by taking the opposite value for the constants in (4.30a):

$$\beta = -e^{-a\omega}e^{-a\psi} \tag{6.21c}$$

$$b = a(s + t)e^{-a\omega}e^{-a\psi} \tag{6.21d}$$

To make the connection with (6.11), we define the combinations

$$\varphi = \omega + \psi \tag{6.22a}$$

$$\tilde{\varphi} = \omega - \psi \tag{6.22b}$$

$$u = s + t \tag{6.22c}$$

$$\tilde{u} = s - t \tag{6.22d}$$

Substituting (6.22) into (6.21), we recover (6.11).

We now turn to the question of whether the spectrum of states is equivalent in the superbosonized representation.[48] Actually the question is easier to answer here than in the bosonic theory, since here both before and after bosonization, the theory posesses two fermionic variables and two bosonic variables. However, the superbosonized fields φ and $\tilde{\varphi}$ both have invariant zero modes. We need to determine the spectrum of eigenvalues for these operators. The superghosts β and γ satisfy boundary conditions of the form $\beta(\sigma + 2\pi) = \pm\beta(\sigma)$ ($-$ in the Neveu-Schwarz sector, $+$ in the Ramond sector). From (6.11d) we see that the eigenvalue spectrum of p_φ which satisfy these conditions is $p_\varphi = i\frac{2n+1}{2a}$ (NS sector) or $i\frac{n}{a}$ (R sector), where n is any integer. Since b and c must be single valued, the modes of u and $\tilde{u}$ will be half-integral in the NS sector and integral in the R sector. Finally, (6.11c) does not put any constraint at this stage on the eigenvalues of the zero mode $p_{\tilde{\varphi}}$. However, (6.13b) shows that $-\frac{i}{2a}p_{\tilde{\varphi}}$ is equal to the superghost number, so its eigenvalues are half-integral in the NS sector and integral in the R sector. The other invariant zero mode p_φ generates a set of eigenstates that are not present in the standard representation. Furthermore, in the superbosonized representation, the zero mode Virasoro generator L_0 has the form

$$L_0^{gh} = \frac{1}{2}\sum_{n>0}\Big(n\tilde{u}_{-n}u_n + nu_{-n}\tilde{u}_n + \varphi_{-n}\tilde{\varphi}_n + \tilde{\varphi}_{-n}\varphi_n\Big) + \frac{1}{2}p_\varphi p_{\tilde{\varphi}} \tag{6.23a}$$

By comparison, with the usual ghosts,

$$L_0 = \sum_{n>0} n\Big(b_{-n}c_n + c_{-n}b_n + \beta_{-n}\gamma_n - \gamma_{-n}\beta_n\Big) \tag{6.23b}$$

Comparing these two expressions, we see that while (6.23b) is bounded from below, (6.23a), because of the term $p_{\varphi}p_{\tilde{\varphi}}$, is not bounded from either direction. Clearly, the space of states is different in the two representations. Some sort of truncation of the spectrum is therefore necessary if we want to have equivalent state spaces. By restricting our attention to those states in the theory which satisfy $p_{\tilde{\varphi}} = 2a^2 p_{\varphi}$, we get in L_0 a term proportional to N_{SG}^2, which makes L_0 bounded from below[48] and agrees with the superghost number dependence of (6.23b).

6.2 Construction of Invariants

Let us now consider the question of what invariant operators exist in the superbosonized theory, in particular, the BRST charge[18,19], which is normally constructed as a product of doublets. Neglecting ordering effects for the moment, we may combine (c, γ) with the doublet $(\mathcal{M}^L(\sigma), 2M^L(\sigma))$ using the rule (4.30a) to form the covariant doublet $(P_{cl}(\sigma), Q_{cl}(\sigma))$. Then we find

$$\begin{aligned} P_{cl} = \Big(&-3a(t-s)\Gamma \cdot x' - \frac{3a}{2} ts\omega' + \frac{3a}{2} st\psi' \\ &+ 3(s-t)\left[(a^2/2 + 3/4)s' - (a^2/2 - 3/4)t'\right]\Big)e^{a\omega}e^{a\psi} \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} Q_{cl} =& -3a(t-s)(x'^2 + \Gamma\Gamma' - \omega'^2 + 2w\omega'' + \psi'^2 - 2v\psi'' - s's + t't)e^{a\omega}e^{a\psi} \\ &+ [\frac{3a}{2}(\omega' - \psi') + 3a^2 st][\Gamma \cdot x' - s\omega' + 2ws' + t\psi' - 2vt']e^{a\omega}e^{a\psi}. \end{aligned} \tag{6.25}$$

Here w and v are the weight parameters of ω and ψ respectively, given (from (6.20) and (6.19)) by

$$w = -(\frac{a}{2} + \frac{3}{4a})$$

and

$$v = (\frac{a}{2} - \frac{3}{4a}).$$

Unfortunately, ordering effects spoil the covariance of these quantities. We can remedy this problem by adding terms to Q in order to make it both nilpotent and truly super-reparametrization invariant. To find all possible invariant operators we will simply write down all possible covariant quantities and try to assemble an operator doublet $(P(\sigma), Q(\sigma))$ for which the integral of the heavier weight component $Q = \int d\sigma Q(\sigma)$ is invariant.[48] It is simplest to first determine the form of the lighter weight component $P(\sigma)$ and transform it to get Q. We write down all possible terms P_i which are weight $\frac{1}{2}$ up to anomalies. The correct P will be some linear combination of these. We then require that P transform as the lighter component of a doublet of the standard form (4.25), and that the anomalies cancel. This is accomplished by demanding that the f'' and f''' terms in $\delta_f P$ and the f' and f'' terms in $\not{\delta}_f P$ add to zero. These restrictions select out a four parameter set of solutions. This calculation is described in Appendix B. The result is

$$Q = AQ_A + BQ_B + CQ_C + DQ_D \tag{6.26}$$

where A,B, C and D are any constants and

$$\begin{aligned} Q_A =& -\frac{9}{d}(2a^2+9)Q_1 + 3aQ_2 + 3aQ_3 - 2aQ_4 \\ & - 4a(9+a^2)Q_5 + 4a^3Q_6 - 2a(27+2a^2)Q_7 + 4a^3Q_8 - 54aQ_9 \end{aligned} \tag{6.27a}$$

$$\begin{aligned} Q_B =& \frac{9}{d}(-2a^2+9)Q_1 - 3aQ_2 - 3aQ_3 + 2aQ_4 \\ & + 4a^3Q_5 + 4a(9-a^2)Q_6 + 4a^3Q_7 + 2a(27-2a^2)Q_8 - 54aQ_{10} \end{aligned} \tag{6.27b}$$

$$
\begin{aligned}
Q_C = &-\frac{18a}{d}(9+2a^2)Q_1 - 3(2a^2-9)Q_2 - 3(27+2a^2)Q_3 \\
&+ 2(27+2a^2)Q_4 + 4a^2(2a^2-9)Q_5 - 4a^2(27+2a^2)Q_6 \\
&+ 4a^2(-27+2a^2)Q_7 - 4a^2(27+2a^2)Q_8 + 54aQ_{11}
\end{aligned} \tag{6.27c}
$$

$$
\begin{aligned}
Q_D = &-\frac{18a}{d}(2a^2-9)Q_1 + 3(-27+2a^2)Q_2 + 3(9+2a^2)Q_3 \\
&- 2(2a^2-27)Q_4 - 4a^2(2a^2-27)Q_5 + 4a^2(9+2a^2)Q_6 \\
&- 4a^2(2a^2-27)Q_7 + 4a^2(27+2a^2)Q_8 + 54aQ_{12}
\end{aligned} \tag{6.27d}
$$

where the Q_i's are defined as follows:

$$Q_1 = -((t-s)(x'^2 + \Gamma'\Gamma) + (\psi' - \omega' + 2ast)x' \cdot \Gamma)e^{a\omega}e^{a\psi} \tag{6.28a}$$

$$Q_2 = -a[s'' + 2as'\omega' + 2s(a\omega'' + a^2\omega'^2)]e^{a\omega}e^{a\psi} \tag{6.28b}$$

$$Q_3 = -a[t'' + 2at'\psi' + 2t(a\psi'' + a^2\psi'^2)]e^{a\omega}e^{a\psi} \tag{6.28c}$$

$$Q_4 = -a^2[s'\psi' + a(s+t)\omega'\psi' + t'\omega']e^{a\omega}e^{a\psi} \tag{6.28d}$$

$$Q_5 = [-at\omega'^2 + ass'\ t + as\omega'\psi']e^{a\omega}e^{a\psi} \tag{6.28e}$$

$$Q_6 = [as\psi'^2 - astt' - at\omega'\psi']e^{a\omega}e^{a\psi} \tag{6.28f}$$

$$Q_7 = [-t\omega'' - ass'\ t + s'\psi']e^{a\omega}e^{a\psi} \tag{6.28g}$$

$$Q_8 = [-t'\omega' + s\psi'' - ast'\,t]e^{a\omega}e^{a\psi} \tag{6.28h}$$

$$Q_9 = [-s\omega'' + s'\omega' + ass'\ t]e^{a\omega}e^{a\psi} \tag{6.28i}$$

$$Q_{10} = [-t\psi'' + t'\psi' - astt']e^{a\omega}e^{a\psi} \tag{6.28j}$$

$$Q_{11} = -[s'' + as\omega'' + at\omega'']e^{a\omega}e^{a\psi} \tag{6.28k}$$

$$Q_{12} = -[t'' + at\psi'' + as\psi'']e^{a\omega}e^{a\psi} \tag{6.28l}$$

We now need to investigate whether any of the invariant Q operators obtained from the above $Q(\sigma)$'s are nilpotent. Consider a general Q of the form

$$Q = \int \frac{d\sigma}{2\pi} \left(2cM^{x,\Gamma}(\sigma) - \gamma \mathcal{M}^{x,\Gamma}(\sigma)\right) + Q_{gh} \tag{6.29}$$

Using the (anti)commutation relations of the $\mathcal{M}$ and and the M operators, $Q^2 = 0$ is equivalent to the conditions[48]

$$\{Q_{gh}, c(\sigma)\} = i\left(\frac{1}{2}\gamma(\sigma)^2 - 2c(\sigma)c'(\sigma)\right) \tag{6.30a}$$

$$[Q_{gh}, \gamma(\sigma)] = i\Big(c'(\sigma)\gamma(\sigma) - 2c(\sigma)\gamma'(\sigma)\Big) \tag{6.30b}$$

$$Q_{gh}^2 = \frac{iD}{4}\int \frac{d\sigma}{2\pi}\Big(\gamma''(\sigma)\gamma(\sigma) + c'''(\sigma)c(\sigma)\Big) \tag{6.30c}$$

The expressions on the right hand side of (6.30a)can be evaluated in terms of the new ghosts; for instance, the first one is

$$i(\frac{1}{2}\gamma^2 - 2cc') = i : \left(\frac{\tilde{\varphi}'^2}{2} - a\tilde{\varphi}'\tilde{u}u - 2\tilde{u}\tilde{u}' - a\tilde{\varphi}'' + 2a^2\tilde{u}'u\right) e^{2a\varphi} :, \tag{6.31}$$

We find that (6.26) satisfies these conditions for

$$A = 1/3 \tag{6.32}$$

$$B = 1 \tag{6.33}$$

$$C = -\frac{(9 + 14a^2)}{24a^3} \tag{6.34}$$

$$D = -\frac{(2a^2 + 3)}{8a^3} \tag{6.35}$$

so that this combination (up to an overall constant) is indeed a nilpotent operator.

We can of course also derive the expression for the BRST charge in terms of the new ghosts by substituting for the ghosts in the old expression for Q and re-doing the normal-ordering. In terms of the old ghosts, we have

$$Q^{gh} = -\frac{1}{2}\int \frac{d\sigma}{2\pi} : \gamma(\gamma b - 2\beta' c - 3\beta c') - c(4c'b - \gamma\beta' - 3\beta\gamma') : \tag{6.36a}$$

The terms in Q expressed in terms of the new ghosts are as follows:

$$\gamma^2 b = (\frac{\tilde{\varphi}'^2 u}{2} + 2au'\tilde{\varphi}' - 2a^2\tilde{u}uu' - au\tilde{\varphi}'')e^{a\varphi} \tag{6.37}$$

$$\gamma\beta' c = \frac{1}{2}(\frac{\varphi'\tilde{\varphi}'\tilde{u}}{2} + \frac{\tilde{u}''}{2} + a\tilde{u}\varphi'')e^{a\varphi} \tag{6.38}$$

$$cc'b = \left(\frac{a\tilde{u}\varphi''}{2} - \frac{a^2\tilde{u}\varphi'^2}{2} + \frac{\tilde{u}\tilde{u}'u}{2} - a\tilde{u}'\varphi'\right) \tag{6.39}$$

$$\gamma\beta c' = \frac{1}{2}(\tilde{u}'' - \frac{\tilde{\varphi}'\tilde{u}'}{a} - \tilde{u}\varphi'\tilde{\varphi}' + \tilde{u}u\tilde{u}' - 2a\tilde{u}\varphi'')e^{a\varphi} \tag{6.40}$$

$$\beta\gamma' c = -\frac{1}{2a}(\tilde{u}\tilde{\varphi}'' + a\tilde{u}\varphi'\tilde{\varphi}' - a\tilde{u}'u\tilde{u})e^{a\varphi} \tag{6.41}$$

All expressions on both sides of this equation are understood to be normal-ordered. For completeness, we give the final form for the integrand of the nilpotent operator Q:

$$\begin{aligned} Q(\sigma) =& 2\tilde{u}e^{a\varphi}M^{x,\Gamma} - (\tilde{\varphi}' - a\tilde{u}u)e^{a\varphi}\mathcal{M}^{x,\Gamma} \\ &- \frac{1}{2}(\frac{1}{2}u\tilde{\varphi}'^2 + 2au'\tilde{\varphi}' - 2a^2\tilde{u}uu' - au\tilde{\varphi}'' - \frac{1}{2}\tilde{u}\varphi'\tilde{\varphi}' \\ &+ \frac{3}{2a}\tilde{u}'\tilde{\varphi}' + \tilde{u}\tilde{u}'u - \frac{3}{2a}\tilde{\varphi}''\tilde{u} - 2\tilde{u}'' + 2a^2\tilde{u}\varphi'^2 + 4a\tilde{u}'\varphi')e^{a\varphi} \end{aligned} \tag{6.42}$$

We note that this differs somewhat from the expression given in ref.46.

As another application of our methods, we consider the construction of the picture-changing operator.[34,17,48] This operator has weight zero and is constructed as the anti-commutator of the BRST charge with the field $\xi(\sigma)$ (see (6.9)). The bosonized field χ can be written in terms of the new fields as (this can be seen from the operator product $\gamma\beta$)

$$\chi' = a\varphi' - \frac{\tilde{\varphi}'}{2a} + \frac{\tilde{u}u}{2} \tag{6.43}$$

so that the relation

$$e^{\chi} = \tilde{u}e^{a\varphi - \tilde{\varphi}/2a} \tag{6.44}$$

holds. Also, the fields ξ' and η can be written as

$$\xi' = \frac{i}{2a}\tilde{u}e^{-\tilde{\varphi}/2a} \tag{6.45}$$

$$\eta = -(2au' + u\tilde{\varphi}')e^{\tilde{\varphi}/2a} \tag{6.46}$$

Since the picture changer has a term of the form

$$e^{\chi}\Gamma \cdot x' = \tilde{u}e^{a\varphi - \tilde{\varphi}/2a}\Gamma \cdot x' \tag{6.47}$$

we can use our method to write down a general weight zero operator with this term in it. We note that it is essential for the picture changing operator to transform without any f' or f'' terms under $\oint_f$, since all amplitudes calculated with it must of course be invariant under super-reparametrizations. We again find a four parameter family of operators, this time of weight zero. The independent solutions are (with the constant $a = 1$ for convenience)

$$\begin{aligned} P_A =& \frac{1}{10}P_1 + P_5 - \frac{1}{3}P_6 + \frac{1}{2}P_7 - \frac{1}{2}P_8 \\ &+ \frac{1}{2}P_9 + \frac{1}{2}P_{10} \end{aligned} \tag{6.48a}$$

$$\begin{aligned} P_B =& P_2 + \frac{7}{4}P_5 - \frac{7}{4}P_6 + \frac{7}{8}P_7 - \frac{21}{8}P_8 \\ &+ \frac{11}{8}P_9 + \frac{21}{8}P_{10} - \frac{1}{4}P_{11} \end{aligned} \tag{6.48b}$$

$$\begin{aligned} P_C =& P_3 + \frac{81}{4}P_5 - \frac{45}{4}P_6 + \frac{81}{8}P_7 - \frac{135}{8}P_8 \\ &+ \frac{81}{8}P_9 + \frac{171}{8}P_{10} + \frac{3}{4}P_{12} \end{aligned} \tag{6.48c}$$

$$\begin{aligned} P_D =& P_4 + 9P_5 - \frac{11}{2}P_6 + \frac{5}{8}P_7 - \frac{15}{4}P_8 \\ &+ \frac{9}{2}P_9 + \frac{33}{4}P_{10} + \frac{3}{8}P_{11} + \frac{3}{8}P_{12} \end{aligned} \tag{6.48d}$$

Here the P_i's are given by

$$P_1 = x' \cdot \Gamma(t - s)e^{-\omega/2}e^{3\psi/2} \tag{6.49a}$$

$$P_2 = (e^{-\omega/2})'' e^{3\psi/2} \tag{6.49b}$$

$$P_3 = e^{-\omega/2}(e^{3\psi/2})'' \tag{6.49c}$$

$$P_4 = (e^{-\omega/2})'(e^{3\psi/2})' \tag{6.49d}$$

$$P_5 = st(e^{-\omega/2})' e^{3\psi/2} \tag{6.49e}$$

$$P_6 = ste^{-\omega/2}(e^{3\psi/2})' \tag{6.49f}$$

$$P_7 = s'te^{-\omega/2}e^{3\psi/2} \tag{6.49g}$$

$$P_8 = st'e^{-\omega/2}e^{3\psi/2} \tag{6.49h}$$

$$P_9 =: ss' : e^{-\omega/2}e^{3\psi/2} \tag{6.49i}$$

$$P_{10} =: tt' : e^{-\omega/2}e^{3\psi/2} \tag{6.49j}$$

$$P_{11} = \omega'' e^{-\omega/2}e^{3\psi/2} \tag{6.49k}$$

$$P_{12} = \psi'' e^{-\omega/2}e^{3\psi/2} \tag{6.49l}$$

Any combination of P_A, P_B, P_C and P_D is of course a weight zero operator. By comparison of coefficients, the usual BRST-invariant picture-changing operator corresponds to the combination

$$X = 10P_A - \frac{9228}{67}P_B + \frac{10568}{603}P_C - \frac{48}{67}P_D \tag{6.50}$$

We do not yet know if other combinations of these four operators exist which are also BRST-invariant. It would be of potential interest to find these, if they do exist, since they would be of particular use in the construction of interactions for superstring field theories.

6.3 Summary

We have shown that a superbosonized representation of the superconformal ghosts in terms of two doublets can be obtained using our algebraic techniques.

The space of states was shown to be larger than the usual one and a correct subspace was identified by means of a suitable constraint. A search for dynamical invariants produced a hitherto unknown four-parameter class of such objects. These new objects, being dynamical invariants, are worthy of further investigation. The picture-changing operator of superstring field theory was identified as one member of a four-parameter class of weight zero operators which change the picture number. The existence of these operators offers interesting possibilities for building interacting superstring field theories.

APPENDIX A
REDUCIBILITY OF THE SUPER-REPARAMETRIZATION REPRESENTATIONS

We shall start with the master equations (4.45) for the transformation of a doublet (a, b) into (A, B). We shall assume, as before, that the bosonic or fermionic character of the light component is left unchanged[36] by the transformation matrix. We recapitulate the master equations here for convenience:

$$\begin{aligned}
(\delta\!\!\!/_f F_{11})a &= -F_{12}(f\frac{d}{d\sigma} + 2w_a f')a - fF_{21}a \\
(\delta\!\!\!/_f F_{12})b &= F_{11}fb - fF_{22}b \\
(\delta\!\!\!/_f F_{21})a &= F_{22}(f\frac{d}{d\sigma} + 2w_a f')a - f(F_{11}a)' - 2w_A f' F_{11}a \\
(\delta\!\!\!/_f F_{22})b &= -F_{21}(fb) - f(F_{12}b)' - 2w_A f' F_{12}b
\end{aligned} \tag{A.1}$$

We expand each of the F's in a finite series of derivative operators:

$$F = \sum G^n \frac{d^n}{d\sigma^n}.$$

Let the highest order derivative operator appearing in the expansion of F_{21} have order N. Then it is easy to see that we have two families of representations. In the first case, the highest derivative operator in the expansions of F_{11} and F_{22} has order N, and that in F_{12} has order $N-1$; in this case we have the constraint that

$$G_{11}^N = G_{22}^N.$$

Thus the representations in this case consist of $(4N+2)$ independent fields (the G's).

In the second case, the highest derivative operator in the expansions of F_{11}, F_{22} and F_{12} has order $N-1$; in this case we have the constraint that

$$G_{21}^N = -G_{12}^{N-1}. \tag{A.2}$$

The representations in this case consist of $4N$ independent fields. Note that this case can be obtained from the previous one by setting $G_{11}^N = G_{22}^N = 0$ and imposing the constraint (A.2).

In either case, we can obtain the equations for the supersymmetry transformations of the G's by equating the coefficients of derivatives of a and b in the equations (A.1). We get the following equations:

$$\begin{aligned}
\not\delta_f G_{11}^r &= -\sum_{n=r-1}^{N} G_{12}^n f^{(n-r+1)} \binom{n}{r-1} - 2w_a \sum_{n=r}^{N} G_{12}^n f^{(n-r+1)} \binom{n}{r} - f G_{21}^r \\
\not\delta_f G_{12}^r &= \sum_{n=r}^{N} G_{11}^n f^{(n-r)} \binom{n}{r} - f G_{22}^r \\
\not\delta_f G_{21}^r &= \sum_{n=r-1}^{N} G_{22}^n f^{(n-r+1)} \binom{n}{r-1} + 2w_a \sum_{n=r}^{N} G_{22}^n f^{(n-r+1)} \binom{n}{r} \\
&\quad - f G_{11}^r{}' - f G_{11}^{r-1} - 2w_A f' G_{11}^r \\
\not\delta_f G_{22}^r &= -\sum_{n=r}^{N} G_{21}^n f^{(n-r)} \binom{n}{r} - f(G_{12}^r{}' + G_{12}^{r-1}) - 2w_A f' G_{12}^r
\end{aligned} \tag{A.3}$$

If we define a generalized covariant derivative operator of order N as

$$\mathcal{O} = \sum_{n=0}^{N} A^n \frac{d^n}{d\sigma^n}$$

and demand that it act on a covariant field F of weight w_F to produce a new covariant field of weight $(w_F + \Delta)$, we can read off the transformations of the A^n from (3.35). They are

$$\delta_f A^r = D_f^{(\Delta - r)} A^r + \sum_{m=r+1}^{N} \left[\binom{m}{r-1} + w_F \binom{m}{r} \right] f^{(m-r+1)} A^m. \tag{A.4}$$

It can be checked that the above transformations on the G's indeed satisfy

$$\not\delta_f \not\delta_f = -\delta_{ff},$$

so that they indeed form a representation of the supersymmetry algebra. Under a reparametrization δ_f, G^r_{11} and G^r_{22} transform with weight $(\Delta - r)$, G^r_{12} with weight $(\Delta - r - \frac{1}{2})$, and G^r_{21} with weight $(\Delta - r + \frac{1}{2})$, apart from anomaly terms which have the same form as in (A.4). Specifically,

$$\begin{aligned}
\delta_f G^r_{11} &= D_f^{(\Delta-r)} G^r_{11} + \sum_{m=r+1}^{N} \left[\binom{m}{r-1} + w_a \binom{m}{r} \right] f^{(m-r+1)} G^m_{11} \\
\delta_f G^r_{12} &= D_f^{(\Delta-r-\frac{1}{2})} G^r_{12} + \sum_{m=r+1}^{N} \left[\binom{m}{r-1} + (w_a + \frac{1}{2}) \binom{m}{r} \right] f^{(m-r+1)} G^m_{12} \\
\delta_f G^r_{21} &= D_f^{(\Delta-r+\frac{1}{2})} G^r_{21} + \sum_{m=r+1}^{N} \left[\binom{m}{r-1} + w_a \binom{m}{r} \right] f^{(m-r+1)} G^m_{21} \\
\delta_f G^r_{22} &= D_f^{(\Delta-r)} G^r_{22} + \sum_{m=r+1}^{N} \left[\binom{m}{r-1} + (w_a + \frac{1}{2}) \binom{m}{r} \right] f^{(m-r+1)} G^m_{22}
\end{aligned} \tag{A.5}$$

We shall now consider the reducibility of these representations. As we have shown earlier, the only irreducible representations in terms of covariant fields are doublets. Our modus operandi shall hence consist of starting from the lowest weight field in the representation (which necessarily is the member of a covariant doublet) and working our way up the weight 'ladder', trying to form a covariant doublet at each stage. The existence of a new doublet at each level implies that the fields in all the previous levels decouple completely from those at this level and at all further levels. To illustrate this procedure, let us first look at the $(4N+2)$-field representation. The lowest weight field in this multiplet is $G^N_{11}(= G^N_{22})$, which has weight $(\Delta - N)$. This transforms as

$$\not{\delta}_f G^N_{11} = -f(G^{N-1}_{12} + G^N_{21}).$$

We look now for a different linear combination of G^{N-1}_{12} and G^N_{21} which transforms as the light component of a doublet(i.e., the transformation does not

involve f'). We find that

$$\oint_f(G_{12}^{N-1}+\alpha G_{21}^{N}) = f\left[(1-\alpha)(G_{11}^{N-1}-G_{22}^{N-1})-\alpha G_{11}^{N\prime}\right]+(N+\alpha N-2\alpha\Delta)f'G_{11}^{N}. \tag{A.6}$$

Requiring that the f' term vanish, we get

$$\alpha = N/(2\Delta - N) \tag{A.7}$$

So the combinations

$$(2\Delta - N)G_{12}^{N-1} + NG_{21}^{N}$$

and

$$2(\Delta - N)(G_{11}^{N-1} - G_{22}^{N-1}) - NG_{11}^{N\prime}$$

form a new doublet, provided that

$$\alpha \neq 1,$$

i.e.,$\Delta \neq N$. If $\Delta = N$, this doublet is the same as before; therefore, there is no reducibility at this level. In this case, as we shall see in a moment, the next level separates out, leaving us with an irreducible quartet at this level. Continuing this process, let us consider

$$\begin{aligned}\oint_f(G_{11}^{N-1} + \beta G_{22}^{N-1} + \gamma G_{11}^{N\prime}) = \\ &- f\left[(1+\beta)(G_{12}^{N-2} + G_{21}^{N-1}) + (\beta+\gamma)G_{12}^{N-1\prime} + \gamma G_{21}^{N\prime}\right] \\ &- f'\left[(N-1+2w_a+2\beta w_A+\gamma)G_{12}^{N-1} + (N\beta+\gamma)G_{21}^{N}\right]\end{aligned} \tag{A.8}$$

This combination transforms without the f' term if we choose

$$\beta = \frac{2w_a + N - 1}{N - 2w_A} \qquad \text{and} \qquad \gamma = \frac{N(N-1+2w_a)}{2w_A - N}. \tag{A.9}$$

Hence the combinations

$$(N - 2w_A)G_{11}^{N-1} + (N - 1 + 2w_a)(G_{22}^{N-1} - NG_{11}^{N\prime})$$

and

$$(2N - 2\Delta - 1)(G_{12}^{N-2} + G_{21}^{N-1}) + (N - 1 + 2w_a)\left((1 - N)G_{12}^{N-1\prime} - NG_{21}^{N\prime}\right)$$

form a new doublet, as long as $\Delta \neq N - \frac{1}{2}$. If $\Delta = N - \frac{1}{2}$, there is no reducibility at this level; so far only the first doublet has decoupled completely. The decomposition of the sextet presented in the third chapter follows this same pattern.

Going a step further up the ladder of weights, we can now look at the transformation of a different linear combination of the fields G_{12}^{N-2}, G_{21}^{N-1}, $G_{12}^{N-1\prime}$ and $G_{21}^{N\prime}$, namely,

$$\not{\delta}_f(G_{12}^{N-2} + \mu G_{21}^{N-1} + \nu G_{12}^{N-1\prime} + \rho G_{21}^{N\prime}). \tag{A.10}$$

This time, however, in addition to the f' term, there is an f'' term; both of these terms must vanish if we want a reduction into doublets. This yields four conditions for the three parameters μ, ν and ρ, which are in general consistent only if the relation

$$(2N - 1 - 2\Delta)(\Delta + 2w_a) - (N - 1 + 2w_a) = 0 \tag{A.11}$$

is satisfied. We note that this relation has $\Delta = N - 1$ as a possible solution; thus, in this case, the representation is reducible at this level. This is in complete contrast to the previous two stages of reduction, where the doublets would decouple except for special values of the weights. If $\Delta = N - 1$, three doublets have by now completely decoupled. For higher levels, the number of constraints increases faster than the number of coefficients in the combinations of fields. Then reducibility breaks down in general, leaving us with larger and larger irreducible(non-covariant) multiplets.

Next we consider the case of the $4N$-field representations. Now the lowest weight field in the multiplet is G_{12}^{N-1}, with weight $(\Delta - N + \frac{1}{2})$ and $G_{11}^{N} = G_{22}^{N} = 0$. We have

$$\oint_f G_{12}^{N-1} = f(G_{11}^{N-1} - G_{22}^{N-1}). \tag{A.12}$$

As before, we consider the transformation of a different linear combination of the fields on the right hand side:

$$\begin{aligned}\oint_f(G_{11}^{N-1} + \alpha G_{22}^{N-1}) = &- f\left((1+\alpha)(G_{12}^{N-2} + G_{21}^{N-1}) + \alpha G_{12}^{N-1'}\right) \\ &- f'(N - 1 + 2w_a - \alpha N + 2\alpha w_A)G_{12}^{N-1}\end{aligned} \tag{A.13}$$

The f' term vanishes if we choose

$$\alpha = \frac{N - 1 + 2w_a}{2w_A - N}.$$

Then the combinations

$$(2w_A - N)G_{11}^{N-1} + (N - 1 + 2w_a)G_{22}^{N-1}$$

and

$$(2w_A + 2w_a - 1)(G_{12}^{N-2} + G_{21}^{N-1}) + (N - 1 + 2w_a)G_{12}^{N-1'}$$

form a new doublet if $\alpha \neq -1$, i.e., if $w_a + w_A \neq \frac{1}{2}$. Let us move on to the next level and look at the f' term in

$$\oint_f(G_{12}^{N-2} + \beta G_{21}^{N-1} + \gamma G_{12}^{N-1'}),$$

which is

$$f'\left((N - 1 + \gamma)G_{11}^{N-1} - \beta(N - 1 + 2w_a)G_{12}^{N-1} - \gamma G_{22}^{N-1}\right).$$

This is zero if and only if $\beta = \gamma = 0$ and $N = 1$, in which case this level does not even exist. As we go up to higher levels, we find as before that there are too many constraints on too few parameters, so that in general, only the first doublet decouples completely from these representations.

APPENDIX B
EXPLICIT CONSTRUCTION OF INVARIANTS

We now show the details of the construction of the operator P whose superpartner integrates to form an invariant Q. Assuming that P is a commuting operator, it may be built as

$$P(\sigma) = \sum_{i=1}^{12} A_i P_i(\sigma) \tag{B.1}$$

where

$$P_1 = x' \cdot \Gamma(t-s) e^{a\omega} e^{a\psi} \tag{B.2a}$$

$$P_2 = (e^{a\omega})'' e^{a\psi} \tag{B.2b}$$

$$P_3 = e^{a\omega} (e^{a\psi})'' \tag{B.2c}$$

$$P_4 = (e^{a\omega})' (e^{a\psi})' \tag{B.2d}$$

$$P_5 = st(e^{a\omega})' e^{a\psi} \tag{B.2e}$$

$$P_6 = ste^{a\omega} (e^{a\psi})' \tag{B.2f}$$

$$P_7 = s'te^{a\omega} e^{a\psi} \tag{B.2g}$$

$$P_8 = st'e^{a\omega} e^{a\psi} \tag{B.2h}$$

$$P_9 =: ss' : e^{a\omega} e^{a\psi} \tag{B.2i}$$

$$P_{10} =: tt' : e^{a\omega} e^{a\psi} \tag{B.2j}$$

$$P_{11} = \omega'' e^{a\omega} e^{a\psi} \tag{B.2k}$$

$$P_{12} = \psi'' e^{a\omega} e^{a\psi} \tag{B.2l}$$

For Q to be super-reparametrization invariant, P must transform with weight $1/2$; since we have $2a(w + a/2) = -3/2$, the exponentials $e^{a\omega} e^{a\psi}$ have

total weight $-\frac{3}{2}$, so these are the only terms P can have. As already stated, ordering effects will alter the super-reparametrization transformations of these quantities. Useful results are

$$\omega'(\sigma_1)e^{a\omega(\sigma_2)} =: \omega'(\sigma_1)e^{a\omega(\sigma_2)} : + \frac{a}{\sigma_1 - \sigma_2} e^{a\omega(\sigma_2)} \tag{B.3a}$$

$$\psi'(\sigma_1)e^{a\psi(\sigma_2)} =: \psi'(\sigma_1)e^{a\psi(\sigma_2)} : - \frac{a}{\sigma_1 - \sigma_2} e^{a\psi(\sigma_2)} \tag{B.3b}$$

$$s(\sigma_1)s(\sigma_2) =: s(\sigma_1)s(\sigma_2) : + \frac{1}{\sigma_1 - \sigma_2} \tag{B.3c}$$

$$t(\sigma_1)t(\sigma_2) =: t(\sigma_1)t(\sigma_2) : - \frac{1}{\sigma_1 - \sigma_2}. \tag{B.3d}$$

These equations hold in the limit where $\sigma_1 - \sigma_2 \to 0$. The exact expressions of course involve periodic functions. To illustrate how the normal ordering affects the transformation properties, we show one of the calculations:

$$\begin{aligned}
\not{\delta}_f(se^{a\omega}) &= \lim_{\sigma_1 \to \sigma} \left[-(f(\sigma)\omega'(\sigma) + 2wf'(\sigma))e^{a\omega(\sigma_1)} - s(\sigma)(-af(\sigma_1)s(\sigma_1)e^{a\omega(\sigma_1)}) \right] \\
&= -f : \omega' e^{a\omega} : -2wf'e^{a\omega} - a \lim_{\sigma_1 \to \sigma} \left(\frac{f(\sigma) - f(\sigma_1)}{\sigma - \sigma_1} \right) e^{a\omega(\sigma_1)} \\
&= -f : \omega' e^{a\omega} : -(2w + a)f'e^{a\omega}
\end{aligned} \tag{B.4}$$

Using (B.3) (and various derivatives thereof), we calculate the following results (all quantities are normal-ordered):

$$\not{\delta}_f P_1 = -f((t-s)(x'^2 + \Gamma'\Gamma) + (\psi' - \omega' + 2ast)x' \cdot \Gamma)e^{a\omega}e^{a\psi} - Df''(t-s)e^{a\omega}e^{a\psi} \tag{B.5a}$$

$$\not{\delta}_f P_2 = -a\left(f[s'' + 2as'\omega' + 2s(a\omega'' + a^2\omega'^2)] + 2f'[s' + as\omega'] + f''s\right) e^{a\omega}e^{a\psi} \tag{B.5b}$$

$$\not{\delta}_f P_3 = -a\left(f[t'' + 2at'\psi' + 2t(a\psi'' + a^2\psi'^2)] + 2f'[t' + at\psi'] + f''t\right) e^{a\omega}e^{a\psi} \tag{B.5c}$$

$$\not{\delta}_f P_4 = -a^2 \left(f[s'\psi' + a(s+t)\omega'\psi' + t'\omega'] + f'[s\psi' + t\omega']\right) e^{a\omega}e^{a\psi} \tag{B.5d}$$

$$\oint_f P_5 = \left(f[-at\omega'^2 + ass'\,t + as\omega'\psi'] + \frac{3}{2}f'(t-s)\omega' - \frac{a}{2}f''t \right) e^{a\omega}e^{a\psi} \quad \text{(B.5e)}$$

$$\oint_f P_6 = \left(f[as\psi'^2 - astt' - at\omega'\psi'] + f'[\frac{3}{2}(t-s)\psi'] - \frac{a}{2}f''s \right) e^{a\omega}e^{a\psi} \quad \text{(B.5f)}$$

$$\oint_f P_7 = \left(f[-t\omega'' - ass'\,t + s'\psi'] + f'[-t\omega' - \frac{3}{2a}s'] + (\frac{3}{2a} + \frac{a}{2})f''t \right) e^{a\omega}e^{a\psi} \quad \text{(B.5g)}$$

$$\oint_f P_8 = \left(f[-t'\omega' + s'\psi'' - ast't] + f'[\frac{3}{2a}t' + s\psi'] + (\frac{a}{2} - \frac{3}{2a})f''s \right) e^{a\omega}e^{a\psi} \quad \text{(B.5h)}$$

$$\oint_f P_9 = \left(f[-s\omega'' + s'\omega' + ass'\,t] - f'[s\omega' - \frac{3}{2a}s'] - (\frac{3}{2a} + \frac{a}{2})f''s \right) e^{a\omega}e^{a\psi} \quad \text{(B.5i)}$$

$$\oint_f P_{10} = \left(f[-t\psi'' + t'\psi' - astt'] - f'[t\psi' - \frac{3}{2a}t'] + (\frac{a}{2} - \frac{3}{2a})f''t \right) e^{a\omega}e^{a\psi} \quad \text{(B.5j)}$$

$$\oint_f P_{11} = -\left(f[s'' + as\omega'' + at\omega''] + 2f's' + f''s \right) e^{a\omega}e^{a\psi} \quad \text{(B.5k)}$$

$$\oint_f P_{12} = -\left(f[t'' + at\psi'' + as\psi''] + 2f't' + f''t \right) e^{a\omega}e^{a\psi} \quad \text{(B.5l)}$$

We also need to investigate the transformation of P under the reparametrizations δ. We find

$$\begin{aligned}
\delta_f P = &-fP' - \frac{1}{2}f'P - f''[(-\frac{1}{2}A_2 - \frac{3}{4}A_4 + \frac{1}{a}A_{11})(e^{a\omega})'e^{a\psi} \\
&+ (-\frac{1}{2}A_3 - \frac{3}{4}A_4 + \frac{1}{a}A_{12})e^{a\omega}(e^{a\psi})' \\
&+ (-\frac{3}{4}A_5 - \frac{3}{4}A_6 + \frac{1}{2}A_7 + \frac{1}{2}A_8)ste^{a\omega}e^{a\psi}] \\
&- f'''[-\frac{3}{4}A_2 - \frac{3}{4}A_3 - \frac{1}{12}A_9 + \frac{1}{12}A_{10} - (\frac{3}{4a} + \frac{a}{6})A_{11} + (\frac{a}{6} - \frac{3}{4a})A_{12}]e^{a\omega}e^{a\psi}
\end{aligned} \quad \text{(B.6)}$$

We require, in order to maintain covariance, that the f'' and f''' terms in $\delta_f P$ and the f' and f'' terms in $\oint_f P$ be zero. The general solution for the constants A_i which satisfy these constraints is

$$P = AP_A + BP_B + CP_C + DP_D \quad \text{(B.7)}$$

where A, B, C and D are any constants and

$$P_A = -\frac{9}{d}(2a^2+9)P_1 + 3aP_2 + 3aP_3 - 2aP_4 \\ - 4a(9+a^2)P_5 + 4a^3P_6 - 2a(27+2a^2)P_7 + 4a^3P_8 - 54aP_9 \tag{B.8a}$$

$$P_B = \frac{9}{d}(-2a^2+9)P_1 - 3aP_2 - 3aP_3 + 2aP_4 \\ + 4a^3P_5 + 4a(9-a^2)P_6 + 4a^3P_7 + 2a(27-2a^2)P_8 - 54aP_{10} \tag{B.8b}$$

$$P_C = -\frac{18a}{d}(9+2a^2)P_1 - 3(2a^2-9)P_2 - 3(27+2a^2)P_3 \\ + 2(27+2a^2)P_4 + 4a^2(2a^2-9)P_5 - 4a^2(27+2a^2)P_6 + 4a^2(-27+2a^2)P_7 \\ - 4a^2(27+2a^2)P_8 + 54aP_{11} \tag{B.8c}$$

$$P_D = -\frac{18a}{d}(2a^2-9)P_1 + 3(-27+2a^2)P_2 + 3(9+2a^2)P_3 \\ - 2(2a^2-27)P_4 - 4a^2(2a^2-27)P_5 + 4a^2(9+2a^2)P_6 - 4a^2(2a^2-27)P_7 \\ + 4a^2(27+2a^2)P_8 + 54aP_{12} \tag{B.8d}$$

The corresponding partners $Q(\sigma)$ can of course be obtained from the transformations of the P's; the integrals of the $Q(\sigma)$'s are then invariants.

We now give a list of other constructions which lead to invariant operators. First, we can use (6.18a) if we choose the parameters such that the product doublet has weight $(\frac{1}{2}, 1)$. This is written schematically as

$$(e^{a\omega}, ase^{a\omega}) \otimes_a (e^{b\psi}, bte^{b\psi}) = (e^{a\omega}e^{b\psi}, (as+bt)e^{a\omega}e^{b\psi}) \tag{B.9}$$

where a and b are chosen so that $a(w+a/2)+b(v-b/2)=\frac{1}{2}$. Secondly we can repeat the process used above to construct (P,Q), but with arbitrary constants a and b such that the total weight is still $(\frac{1}{2}, 1)$. We must redefine

$$P_1 = x' \cdot \Gamma[(v-b/2)s - (w+a/2)t]e^{a\omega}e^{b\psi} \tag{B.10}$$

and change the exponentials $e^{a\psi}$ to $e^{b\psi}$ in the other P's. We can then follow a procedure identical to the one described above to make a doublet (P, Q). The analysis follows closely the previous case, and we will give the result only for the particular case where $A_{11} = A_{12} = 0$. Any operator will be invariant which is a linear combination of P_A and P_B, where

$$\begin{aligned}
P_A =& \frac{4ab}{d}(w + a/2)(a - 6(w + a/2))P_1 + b(v - b/2)(1 + 2b(v - b/2))P_2 \\
& - 2a(w + a/2)(1 + b(v - b/2))P_3 + 2(1 + b(v - b/2))(1 + 2b(v - b/2))P_4 \\
& + b(a + 2ab(v - b/2) + 6(w + a/2))P_5 + 2ab(1 + b(v - b/2))P_6 \\
& + ab(1 + 2b(v - b/2) - 12(w + a/2)^2)P_7 \\
& + 2ab(1 + b(v - b/2))P_8 - 12ab(w + a/2)(v - b/2)P_9
\end{aligned} \tag{B.11a}$$

$$\begin{aligned}
P_B =& \frac{4ab}{d}(v - b/2)(b + 6(v - b/2))P_1 - b(v - b/2)(1 + 2b(v - b/2))P_2 \\
& + 2a(w + a/2)(1 + b(v - b/2))P_3 - 2(1 + b(v - b/2))(1 + 2b(v - b/2))P_4 \\
& - ab(1 + 2b(v - b/2))P_5 - 2a(b + 3(v - b/2) + b^2(v - b/2))P_6 \\
& - ab(1 + 2b(v - b/2))P_7 \\
& - 2ab(1 + b(v - b/2) - 6(v - b/2)^2)P_8 - 12ab(w + a/2)(v - b/2)P_{10}
\end{aligned} \tag{B.11b}$$

The expression for the Q's can be obtained by taking the super-reparametrization transformation of (B.11). For these operators to be invariant, it is necessary that $a(w+a/2)+b(v-b/2) = -\frac{3}{2}$. It can be checked that with the restrictions $a = b$ and $w + a/2 = v - b/2$, the operators P_A and P_B of the last section are obtained. It is also possible to generalize this type of invariant to Lorentz tensors. This is done simply by redefining the P's once again:

$$P_1^{\mu\nu} = x'^{\mu}\Gamma^{\nu}[b(v - b/2)s - a(w + a/2)t]e^{a\omega}e^{b\psi} \tag{B.12a}$$

$$P_i^{\mu\nu} = \frac{1}{10}g^{\mu\nu}P_i,\ i > 1 \tag{B.12b}$$

Then we may once again follow the same procedure and construct the invariants as before. Finally, there is the possibility of constructing still more invariants by, for example, combining (6.18a) with $(\mathcal{M}, 2M)$ according to (4.30a) and adjusting the coefficients to cancel the non-invariance caused by ordering effects, if possible. The list of invariants is not exhausted.

To conclude this appendix, we briefly describe how the same process can be used to obtain weight zero objects that change the picture number. As mentioned in the last section, the usual picture-changer contains a term of the form $e^{\chi} = \tilde{u}e^{a\varphi-\tilde{\varphi}/2a}$. The exponential $e^{a\varphi-\tilde{\varphi}/2a}$ can be rewritten as $e^{(a-1/2a)\omega}e^{(a+1/2a)\psi}$ or as $e^{k\omega}e^{l\psi}$ with $k = (a - 1/2a)$ and $l = (a + 1/2a)$. Then the same procedure as above can be employed to get the solutions (with the choice $a = 1$) (6.48).

REFERENCES

1. G. Veneziano, Nuovo. Cim. 57A, 190 (1968).

2. Y. Nambu, Lectures at the Copenhagen Symposium (1970); T. Goto, Prog. Theor. Phys. 46, 1560 (1971).

3. J. Scherk, Rev of Mod. Phys. 47, 123 (1975).

4. *Superstrings*, ed. J.H. Schwarz, World Scientific, Singapore (1985), and references therein.

5. J. Scherk, J. Schwarz, Nucl. Phys. B81, 118 (1974).

6. T. Yoneya, Prog. Theor. Phys. 51, 1907 (1974).

7. D.J. Gross, J. Harvey, E. Martinec, R. Rohm, Nucl. Phys. B256, 253 (1985).

8. C. Lovelace, Phys. Lett. 34B, 500 (1971).

9. R.C. Brower, Phys. Rev. D6, 1655 (1972); P. Goddard, C.B. Thorn, Phys. Lett. 40B, 235 (1972).

10. P. Goddard, J. Goldstone, C. Rebbi, C.B. Thorn, Nucl. Phys. B56, 109 (1973).

11. P. Candelas, G. Horowitz, A. Strominger, E. Witten, Nucl. Phys. B258, 46 (1985).

12. K. Kikkawa, B. Sakita and M.A. Virasoro, Phys. Rev. 184, 1701 (1969); K. Bardakci, M.B. Halpern and J.A. Shapiro, Phys. Rev. 185, 1910 (1969); D. Amati, C. Bouchiat and J.L. Gervais, Nuovo. Cim. Lett. 2, 399 (1969); C. Lovelace, Phys. Lett. 32B, 703 (1970); V. Alessandrini, D. Amati, M. LeBellac, D. Olive, Phys. Rep. 1, 269 (1971); S. Mandelstam, Nucl. Phys. B64, 205 (1973); Phys. Lett. 46B, 447 (1973); Nucl. Phys. B69, 77 (1974).

13. J.A. Shapiro, Phys. Rev. D5, 1945 (1972).

14. A. Sen, Phys. Rev. Lett. 55, 1846 (1985); Phys. Rev. D32, 2102 (1985).

15. M. Kaku, K. Kikkawa, Phys. Rev. D10, 1823 (1974).

16. C. Marshall, P. Ramond, Nucl. Phys. B85, 375 (1975).

17. E. Witten, Nucl. Phys. B268, 79 (1986); Nucl. Phys. B276, 291 (1986).

18. M. Kato, K. Ogawa, Nucl. Phys. B212, 443 (1983).

19. W.Siegel, Phys. Lett. 151B, 396 (1985).

20. T. Banks, M. Peskin, Nucl. Phys. B264 515 (1986).

21. W. Siegel, B. Zwiebach, Nucl. Phys. B263 105 (1986).

22. P. Ramond, V.G.J. Rodgers, R.R. Viswanathan, Nucl. Phys. B293, 293 (1987).

23. P. Ramond, Institute for Fundamental Theory preprint UFIFT-HET-89-5; G. Kleppe. P. Ramond, R.R. Viswanathan, in preparation.

24. D. Pfeffer, P. Ramond, V.G.J. Rodgers, Nucl. Phys. B276, 131 (1986).

25. A. Neveu, P. West, Nucl. Phys. B268, 125 (1986).

26. P. Ramond, Prog. Theor. Phys. 86, 126 (1986).

27. E.C.G. Stückelberg, Helv. Phys. Acta 11, 225 (1938).

28. A. Neveu, J.H. Schwarz, P. West, Phys. Lett. 164B, 51 (1985).

29. A. Ballestrero, E. Maina, Phys. Lett. 180B, 53 (1986).

30. C. Batlle, J. Gomis, Barcelona preprint, UB-FTFP-9/86.

31. S.P. de Alwis, N. Ohta, Phys. Lett. 174B, 388 (1986); Phys. Lett. 188B, 425 (1987).

32. P. Ramond, in *Proceedings of the First Johns Hopkins Workshop*, Jan. 1974, G. Domokos and S. Kövesi-Domokos, eds.

33. C.R.R. Smith,J.G. Taylor, Phys. Lett. 183B, 47 (1987).

34. D.Friedan, E.Martinec, S.Shenker, Nucl. Phys. B271, 93 (1985).

35. G. Kleppe, P. Ramond, R.R. Viswanathan, Phys. Lett. 206B, 466 (1988).

36. G. Kleppe, P. Ramond, R.R. Viswanathan, Nucl. Phys. B315, 79 (1989).

37. J. Weis, unpublished.

38. M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory*, Cambridge University Press, New York (1987), and references therein.

39. M.E. Peskin, in *Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics* (University of California, Santa Cruz, 1986), Howard E. Haber, ed., World Scientific, Singapore (1987), and references therein.

40. A. A. Tseytlin, Phys. Lett. 168B, 63 (1986).

41. G. Müenster, unpublished, DESY 86-045, April 1986.

42. L.Brink, P.Di Vecchia, P.Howe, Phys. Lett. 65B, 471 (1976).

43. S.Deser, B.Zumino, Phys. Lett. 65B, 369 (1976).

44. P.A.Collins, R.W.Tucker, Nucl. Phys. B121, 307 (1977).

45. C. Thorn, Nucl. Phys. B286, 61 (1987).

46. E. Martinec, G. Sotkov, Phys. Lett. 208B, 249 (1988).

47. M. Takama, Phys. Lett. 210B, 153 (1988).

48. G. Kleppe, R.R. Viswanathan, Institute for Fundamental Theory Preprint UFIFT-HEP-89-8, submitted to Nucl. Phys. B.

49. C. B. Thorn, University of Florida preprint UFTP 88-12.

50. G. T. Horowitz, S. Martin, R. C. Myers, Princeton University preprint PUPT-1110 and ITP preprint ITP 88-112.

BIOGRAPHICAL SKETCH

Raju R. Viswanathan was born on July 29, 1963 in New Delhi, India. After completing high school in Bangalore, he went to the Indian Institute of Technology(Madras) in 1979 for his bachelor's degree in chemical engineering. His interest shifted strongly to physics during the course of his study there, and so he came to the University of Florida in August, 1984, for his graduate study in physics, soon after the completion of his undergraduate degree. He became interested in superstring theories as a possible means of description of fundamental interactions. His research in Florida has mainly focused on closed string field theory and the role of reprametrizations as a fundamental invariance in string field theory.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

P. Ramond

Pierre Ramond, Chairman
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Richard D. Field
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Pierre Sikivie
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Charles B. Thorn
Professor of Physics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David A. Drake

David Drake
Professor of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Physics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 1989

Dean, Graduate school